UNIVERSITÄT REGENSBURG
FAKULTÄT FÜR MATHEMATIK

Bachelorarbeit

Tropical Hypersurfaces of Laurent Polynomials

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Deutsche Zusammenfassung

Das Ziel dieser Bachelorarbeit ist ein grundsätzliches Verständnis von *tropischen Hyperflächen*, die von Laurentpolynomen in mehreren Unbekannten mit Koeffizienten aus einem bewerteten Körper erzeugt werden. Wir betrachten die Theorie von bewerteten Körpern und Initialformen, sowie Tropikalisierungen von Laurentpolynomen, Varietäten und Monomabbildungen im algebraischen Torus. Dabei setzen wir kein Hintergrundwissen der algebraischen Geometrie voraus.
Das Hauptresultat dieser Arbeit besteht im Beweis des Satzes von Kapranov, der eine erstaunliche Brücke zwischen der klassischen Varietät eines Laurentpolynoms und dessen tropischer Hyperfläche schlägt. Unter anderem besagt er, dass für einen algebraisch abgeschlossenen, bewerteten Körper K und ein Laurentpolynom f mit Koeffizienten in K die tropische Hyperfläche $\mathrm{trop}(V(f))$ identisch ist mit dem euklidischen Abschluss der Menge der punktweisen Bewertungen der klassischen Varietät $V(f)$.

Das Material dieser Arbeit entstammt größtenteils [MS15], insbesondere folgt der Aufbau Teilen der Kapitel §2.1, §2.2, §2.4, §2.6, §3.1 aus [MS15] mit einigen Änderungen, Zusätzen und gelegentlich detaillierteren Beweisen. Einige Resultate wie beispielsweise Proposition 2.7 und das Beispiel in Bemerkung 2.13 entstammen Übungsaufgaben aus dem Buch. Für die Theorie von Erweiterungen von Bewertungen in Kapitel 2 werden wir teilweise auf zahlentheoretische Ergebnisse wie in [Ne99] verweisen oder einen verallgemeinerten Bewertungsbegriff wie in [EP05] einführen.

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1

Introduction

The study of tropical geometry is the study of geometry over the *tropical semiring* $(\mathbb{R}\cup\{\infty\}, \oplus, \odot)$ in which addition is given by taking the minimum $a \oplus b = \min\{a, b\}$ and multiplication by taking the usual sum $a \odot b = a + b$. Indeed this semiring satisfies all field axioms except for existence of additive inverses – the neutral element of addition is ∞ which represents infinity, and 0 is the neutral element of the multiplicative group. In the tropical semiring, a polynomial with coefficients in $\mathbb{R}$ becomes a piecewise-linear concave function with integer coefficients. For example the tropical version of $f(X) = X^3 + 2X^2 + 4X + 10$ is given by

$$\operatorname{trop}(f)(X) = X^3 \oplus 2 \odot X^2 \oplus 4 \odot X \oplus 10 = \min\{3X, 2X + 2, X + 4, 10\}$$

whose graph is sketched in Figure 1.1. In the case of $\operatorname{trop}(f)$, the two points in the graph for $X = 2$, $X = 6$ indicate where $\operatorname{trop}(f)$ is non-linear and are called *roots* of the polynomial.
The basic objects of study in algebraic geometry are *algebraic varieties* which are defined by zero sets of multivariate polynomials. The corresponding tropical object is the *tropical variety*: For a single tropical polynomial $\operatorname{trop}(f)$ it is defined as exactly the locus in $\mathbb{R}^n$ where the piecewise linear function $\operatorname{trop}(f)$ fails to be linear.

The aim of this bachelor thesis is to present the basic conception of tropical hypersurfaces of multivariate Laurent polynomials over a valued field, as well as all the prerequisites needed. Without requiring background in algebraic geometry, we will study valuations, initial forms and tropicalizations of Laurent polynomials, of varieties and of monomial maps in the algebraic torus. The main result in this

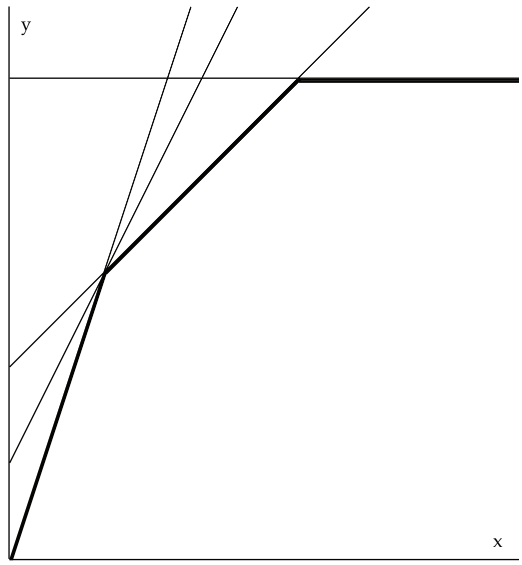

FIGURE 1.1: Graph of trop(f) (indicated by bold line).

thesis is Kapranov's Theorem which draws an astonishing connection between the classical variety defined by a Laurent polynomial and its tropical hypersurface.
The material in this thesis is taken largely from [MS15], specifically it follows parts of §2.1, §2.2, §2.4, §2.6, §3.1 in [MS15] with some alterations, additions and of course more detailed proofs. Some proofs of minor results as for example Proposition 2.7 and the example in Remark 2.13 stem from exercises in the book. For some theory in Chapter 2 about valuations we will refer to number theoretical results as in [Ne99] or to a more generalized theory of valued fields as in [EP05].

Chapter 2 discusses the theory of valued fields. We define valuations from the unit group of a field onto an additive subgroup of $\mathbb{R}$ and look at basic properties of such a map. As algebraically closed, valued fields play the major role throughout this thesis, we focus on how valuations behave on algebraically closed fields. The main results of Chapter 2 are that an arbitrarily valued field can be embedded into an algebraically closed field which extends the valuation (which we will not prove rigorously though), and that in the algebraically closed setting the valuation map always splits, which is important for the definition of *initial forms* in Chapter 4. We also briefly look at important examples of valuations as on the field of Puiseux series or the p-adic valuation. Main source for Chapter 2 is [MS15, §2.1].

In Chapter 3 we outline fundamental objects from algebraic geometry, such as the n-dimensional algebraic torus, the concept of varieties and the Zariski topology. We also introduce monomial maps on the algebraic torus and the maps they induce on the corresponding rings of Laurent polynomials. Chapter 3 consists mainly of various parts of [MS15, §2.2].

Chapter 4 consists of important preliminary work for Kapranov's Theorem. By replacing the coefficients with their valuations and performing all additions and multiplications in the tropical semiring, we can pass from a Laurent polynomial over a valued field to its *tropicalization* – a piecewise linear function from $\mathbb{R}^n$ to $\mathbb{R}$. We define *initial forms* which are special multivariate polynomials with coefficients in the residue class field of a valued field (with respect to its valuation). Initial forms play a distinguished role in this thesis, as they originate from tropicalizations of Laurent polynomials and carry useful information about the roots of the tropical polynomials they come from. Hence Chapter 4 focuses on understanding the basic structure of initial forms and the ideals they generate. At the end of Chapter 4 we briefly talk about the notion of a *tropical basis* and tropicalize monomial maps on the algebraic torus which provides a useful tool for the proof of Kapranov's Theorem.
The material in this chapter is a mixture of [MS15, §2.4] and [MS15, §2.6]; some results will be omitted, some proofs necessarily altered. In general everything will be proved directly for the Laurent polynomial ring and we will try to fill arguments with much detail.

Chapter 5 follows [MS15, §3.1] and is dedicated fully to a detailed proof of Kapranov's Theorem, the main result of this thesis. We define the tropical hypersurface $\mathrm{trop}(V(f))$ of a Laurent polynomial f as the set of all elements $w \in \mathbb{R}^n$ where the minimum in the corresponding tropicalization $\mathrm{trop}(f)(w)$ is achieved at least twice. The main statement of Kapranov's Theorem is then that over an algebraically closed, valued field $\mathrm{trop}(V(f))$ is the same as if one would first compute pointwise the valuation of the classical hypersurface $V(f)$ given by f and afterwards take the Euclidean closure of the resulting set.

2

Fields With Valuations

Definition 2.1. A *valued field* is a pair (K, val) of a field K and a *valuation* map $\mathrm{val}\colon K \to \mathbb{R} \cup \{\infty\}$ satisfying the following properties for all $a, b \in K$:

i.) $\mathrm{val}(a) = \infty \iff a = 0$

ii.) $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$

iii.) $\mathrm{val}(a + b) \geq \min\{\mathrm{val}(a), \mathrm{val}(b)\}$

We will often just say "valued field K" for a pair (K, val). Throughout this thesis, we will also often identify val with its restriction $\mathrm{val}\colon K^* \to \mathbb{R}$. Its image Γ_{val} is an additive subgroup of $\mathbb{R}$, the *value group* of the valued field K. There is a *trivial valuation* on every field K, defined by $\mathrm{val}(a) = 0$ for all $a \in K^*$. Given any valuation val on K and $\lambda \in \mathbb{R}_{>0}$, the map $(\lambda \cdot \mathrm{val})\colon K \to \mathbb{R} \cup \{\infty\}$ is a valuation as well. Thus, for any nontrivial valuation, we may always assume that $1 \in \Gamma_{\mathrm{val}}$.

We consider now the set of all field elements with nonnegative valuation, as well as its subset of elements with positive valuation:

$$R := \{c \in K \mid \mathrm{val}(c) \geq 0\}, \ \mathfrak{m} := \{c \in K \mid \mathrm{val}(c) > 0\}$$

The set R is a ring. It is easy to see that $R \setminus \mathfrak{m} = R^*$, in particular R is a local ring with unique maximal ideal $\mathfrak{m}$. We call R the *valuation ring*, and $\Bbbk := {}^{R}\!/_{\mathfrak{m}}$ its *residue field*.

Remark 2.2. The notion of a valuation can be generalized by defining it as a surjective map

$$\mathrm{val}\colon K \twoheadrightarrow \Gamma \cup \{\infty\}, \tag{2.1}$$

where Γ is a totally ordered abelian group and which satisfies the same properties as in 2.1.
Using Definition 2.1, by property ii.) we see that $\mathrm{val}(1) = 0$ and thus for any $a \in K^* \setminus R$ we must have $a^{-1} \in R$. So the valuation ring R in the sense of Definition 2.1 is also a valuation ring of K in the sense that all $a \in K^*$ must satisfy $a \in R$ or $a^{-1} \in R$. Conversely, every valuation ring in K determines a valuation on K as defined by (2.1) above. Indeed, let R be a valuation ring of K and let Γ be the quotient group $\Gamma := {}^{K^*}/_{R^*}$. This is a totally ordered abelian group by the well-defined relation

$$xR^* \leq yR^* \iff \frac{y}{x} \in R,$$

and the map $\mathrm{val}(x) = xR^* \in \Gamma$ which sends an element $x \in K^*$ to its coset in Γ, defines a valuation on K.

Even further, valuation rings in K correspond one-to-one to valuations on K as in (2.1) up to an order-preserving isomorphism of the value groups. Now an ordered abelian group Γ has *rank 1* – which means that $\{0\}$ is the only proper convex subgroup – if and only if there is an order-preserving isomorphism to a nontrivial subgroup of $(\mathbb{R}, +)$ with the canonical ordering induced from $\mathbb{R}$.
For rigorous proofs of these statements and detailed definitions of the terms used see [EP05, Section 2.1]. Using this generalized theory of valuations and the axiom of choice, the following theorem can be shown. For the full theory, see [EP05, Section 3.1, Section 3.2].

Theorem 2.3. Let $(K, \mathrm{val}_K\colon K \to \mathbb{R} \cup \{\infty\})$ be a valued field, and L/K an algebraic field extension. Then there exists a valuation $\mathrm{val}_L\colon L \to \mathbb{R} \cup \{\infty\}$ on L such that $\mathrm{val}_L|_K = \mathrm{val}_K$.

Remark 2.4. In general, such extensions of valuations will not be unique, see for example 2.14.
As we will notice in Chapter 4, a prerequesite for the definition of *initial forms* is that the valuation map of a valued field K splits. We will also see later that this always happens if K is algebraically closed. Hence Theorem 2.3 plays an important role: If the valuation on K does not split, it is possible to extend

the given valuation to the algebraic closure of K. The problem of extending a valuation on K to some algebraically closed field L with $K \subseteq L$ can also be solved by number theoretical means:
Let $0 < q < 1 \in \mathbb{R}$, then val induces a non-archimedean absolute value

$$|\cdot|_{\mathrm{val},q} \colon K \to \mathbb{R}_{\geq 0}, x \mapsto |x|_{\mathrm{val},q} := q^{\mathrm{val}(x)},$$

where we formally set $q^\infty = 0$. We then pass to the completion L of K with respect to $|\cdot|_{\mathrm{val},q}$ and extend the obtained absolute value on L to an absolute value $\|\cdot\|$ on the algebraic closure $\overline{L}$. This is possible as absolute values on complete fields can be extended to any algebraic extension (for a proof, see e.g. [Ne99, Chapter II. Theorem (4.8)]). This again induces a valuation

$$\mathrm{val}_{\overline{L}} \colon \overline{L} \to \mathbb{R} \cup \{\infty\}, x \mapsto \log_q(\|x\|),$$

where we set $\log_q(0) = \infty$, and which satisfies $\mathrm{val}_{\overline{L}}|_K = \mathrm{val}$.

Lemma 2.5. Let K be valued field and $a, b \in K$. If $\mathrm{val}(a) \neq \mathrm{val}(b)$, then $\mathrm{val}(a+b) = \min\{\mathrm{val}(a), \mathrm{val}(b)\}$.

Proof: Without loss of generality we may assume $\mathrm{val}(b) > \mathrm{val}(a)$. As $\mathrm{val}(1) = 0$, we also have $\mathrm{val}(-1) = 0$, thus $\mathrm{val}(-b) = \mathrm{val}(b)$ for all $b \in K$. We then get

$$\mathrm{val}(a) \geq \min\{\mathrm{val}(a+b), \mathrm{val}(-b)\} = \min\{\mathrm{val}(a+b), \mathrm{val}(b)\},$$

therefore by assumption $\mathrm{val}(a) \geq \mathrm{val}(a+b)$. But also have

$$\mathrm{val}(a+b) \geq \min\{\mathrm{val}(a), \mathrm{val}(b)\} = \mathrm{val}(a),$$

and equality follows. □

The next lemma leads to the result that the residue field of an algebraically closed, valued field is again algebraically closed (see [MS15, Exercise 2.7(4)]).

Lemma 2.6. Let K be valued field. The corresponding valuation ring $(R, \mathfrak{m})$ is integrally closed.

Proof: Let $x \in K = \mathrm{Quot}(R)$ satisfy the monic equation $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$, for $a_0, \ldots, a_{n-1} \in R$.

Suppose $x \notin R$, then (as R is valuation ring) $\mathrm{val}(x^{-1}) > 0$, i.e. $x^{-1} \in \mathfrak{m}$. Multiplying the equation by x^{-n} yields

$$\underbrace{a_{n-1}x^{-1} + \cdots + a_1 x^{-n+1} + a_0 x^{-n}}_{\in \mathfrak{m}} = -1 \in \mathfrak{m},$$

hence $\mathrm{val}(-1) > 0$, a contradiction. □

The following result is a direct consequence of the previous lemma.

Proposition 2.7. Let K be valued field with corresponding valuation ring $(R, \mathfrak{m})$. If K is algebraically closed, then its residue field $\Bbbk$ is algebraically closed as well.

Proof: Let $f(X) = \sum_{i=0}^{n} \overline{a_i} X^i \in \Bbbk[X]$ with coefficients $\overline{a_i} \in \Bbbk$ and appropriate lifts $a_i \in R$. We may assume that f is monic, i.e. $a_n = 1$. As K is algebraically closed, there exists a root $c \in K$ of monic polynomial $g(X) = \sum_{i=0}^{n} a_i X^i \in K[X]$. By 2.6, the root c already lies in R and satisfies

$$f(\overline{c}) = \sum_{i=0}^{n} \overline{a_i c^i} = \overline{\sum_{i=0}^{n} a_i c^i} = 0.$$

□

We will now take a look at two important examples of valuations.

Example 2.8. Let A be a Dedekind domain with quotient field K and $\mathfrak{p} \subseteq A$ a prime ideal. Let $x \in K^*$. The fractional ideal $x \cdot A$ admits a unique decomposition into product of prime ideals

$$x \cdot A = \prod_{\mathfrak{q} \in \mathrm{Spec}(A)} \mathfrak{q}^{v_{\mathfrak{q}}(x)}, \text{ with } v_{\mathfrak{q}}(x) \in \mathbb{Z} \text{ and } v_{\mathfrak{q}}(x) = 0 \text{ for almost all } \mathfrak{q} \in \mathrm{Spec}(A).$$

Hence $\mathfrak{p}$ defines a valuation $\mathrm{val}\colon K \to \mathbb{Z} \cup \{\infty\}$ given by $\mathrm{val}(x) = v_{\mathfrak{p}}(x)$ for $x \in K^*$. For $A = \mathbb{Z}$, $K = \mathbb{Q}$ and $\mathfrak{p} = (p)$, where p is a prime number, this is called the *p-adic valuation* on the rational numbers. For example for the 2-adic valuation get $\mathrm{val}(-\frac{5}{24}) = -3$.

Example 2.9. Given a field K, the field $K\{\{t\}\}$ of *Puisex series* over K is the set of formal series of the form

$$f(t) = \sum_{i=k}^{\infty} c_i t^{i/n}, \text{ where } c_i \in K, n \in \mathbb{N}_{>0}, k \in \mathbb{Z}.$$

We can also consider $K\{\{t\}\}$ as the union

$$K\{\{t\}\} = \bigcup_{n\geq 1} K(\!(t^{1/n})\!),$$

where $K(\!(t^{1/n})\!)$ is the field of formal Laurent series in the variable $t^{1/n}$ (note that in a formal Laurent series, $c_n = 0$ for all but finitely many negative indices n). The field of Puiseux series carries a natural valuation given by taking a nonzero element $f(t) \in K\{\{t\}\}^*$ to the lowest exponent i/n that appears in the series expansion of $f(t)$. Also note that $K(t)$ is a proper subfield of $K(\!(t)\!) \subseteq K\{\{t\}\}$. For example $g(t) = \frac{1}{t^2+5t} \in \mathbb{C}(t)$ has Laurent series expansion

$$\sum_{n=-1}^{\infty} \frac{(-1)^{n+1}}{5^{n+2}} t^n, \text{ thus } \operatorname{val}(g(t)) = -1.$$

An important property of $K\{\{t\}\}$ is that it is algebraically closed if K is algebraically closed and $\operatorname{char}(K) = 0$ (for a proof see [MS15, Thm. 2.1.5]). To be precise, in this case $K\{\{t\}\}$ is the algebraic closure of the Laurent series field $K(\!(t)\!)$ (for a proof see for example [Ri99, 7.1 A.(β)]).

Lemma 2.10. Let K be an algebraically closed field with a nontrivial valuation. Then the value group Γ_{val} is a divisible, dense subgroup of $\mathbb{R}$.

Proof: We first show that Γ_{val} is divisible, i.e. we have to show that for all $a \in \Gamma_{\text{val}}$ and $n \in \mathbb{N}_{>0}$ exists $b \in \Gamma_{\text{val}}$ with $n \cdot b = a$. Let $\operatorname{val}(a) \in \Gamma_{\text{val}}$ for $a \in K^*, n \geq 1$. As K is algebraically closed, there is an element $b \in K$ satisfying $b^n = a$, hence $\operatorname{val}(b) = \frac{1}{n}\operatorname{val}(a)$ and divisibility follows. In addition, as remarked in 2.1, we may assume $1 \in \Gamma_{\text{val}}$, thus $\mathbb{Q} \subseteq \mathbb{R}$ which implies that Γ_{val} is dense in $\mathbb{R}$. □

A crucial property of valuations of algebraically closed, valued fields is that the surjective valuation map splits. Later this will play a key role in the definition of initial forms.

Proposition 2.11. Let K be an algebraically closed, valued field. The surjection $\operatorname{val}\colon K^* \twoheadrightarrow \Gamma_{\text{val}}$ splits, i.e. there is a group homomorphism $\psi\colon (\Gamma_{\text{val}}, +) \to (K^*, \cdot)$ with $\operatorname{val}(\psi(w)) = w$ for all $w \in \Gamma_{\text{val}}$.

Proof: As K is algebraically closed we have $a^{1/n} \in K$ for all $a \in K$ and $n \in \mathbb{Z} \setminus \{0\}$. Thus for all $a \in K^*$ we get a group homomorphism

$$\varphi_a \colon (\mathbb{Q}, +) \to (K^*, \cdot), \quad x \mapsto a^x, \tag{2.2}$$

which obviously satisfies $\mathrm{val}(\varphi_a(x)) = x \cdot \mathrm{val}(a)$. By Lemma 2.10, Γ_{val} is a divisible group, so for all $g \in \Gamma_{\mathrm{val}}$ and $n \in \mathbb{Z} \setminus \{0\}$ there exists $h \in \Gamma_{\mathrm{val}}$ such that $nh = g$. As Γ_{val} is torsion-free as a subgroup of $\mathbb{R}$, this h is unique and we denote the element $\frac{1}{n} \cdot g := h$.
Thus we get a well-defined scalar multiplication

$$\cdot \colon (\mathbb{Q}, \Gamma_{\mathrm{val}}) \to \Gamma_{\mathrm{val}}, \quad \left(\frac{m}{n}, g\right) \mapsto \frac{1}{n} \cdot (mg),$$

which in turn defines on Γ_{val} the structure of a $\mathbb{Q}$-vector space.
Let $(w_i)_{i \in I}$ be a $\mathbb{Q}$-basis of Γ_{val} for an appropriate index set I and recall the canonical isomorphism

$$\phi \colon \Gamma_{\mathrm{val}} \xrightarrow{\sim} \bigoplus_{i \in I} \mathbb{Q}, \quad \sum_{i \in I} \lambda_i w_i \mapsto (\lambda_i)_{i \in I}, \ \mathbb{Q} \ni \lambda_i = 0 \text{ for almost all } i. \tag{2.3}$$

For every $i \in I$ choose $a_i \in K^*$ with $\mathrm{val}(a_i) = w_i$. For the i-th direct summand Γ_i (which is isomorphic to $\mathbb{Q}$), we get a group homomorphism $\varphi_{a_i} \colon \Gamma_i \to K^*$ as defined in (2.2). Furthermore denote by ι_i the canonical inclusion $\iota_i \colon \Gamma_i \to \bigoplus_{i \in I} \mathbb{Q}$. By universal property of direct sum, there is a homomorphism $\tilde{\psi} \colon \bigoplus_{i \in I} (\mathbb{Q}, +) \to (K^*, \cdot)$, such that for all $i \in I$ have $\tilde{\psi} \circ \iota_i = \varphi_{a_i}$, i.e. the lower triangle of diagram (2.4) commutes.

$$\begin{tikzcd}
(\Gamma_{\mathrm{val}}, +) \arrow[r, "\phi"] & \bigoplus_{i \in I} (\mathbb{Q}, +) \arrow[d, "\tilde{\psi}"] \\
\Gamma_i \arrow[ur, "\iota_i"] \arrow[r, "\varphi_{a_i}"'] & (K^*, \cdot)
\end{tikzcd} \tag{2.4}$$

The homomorphism $\tilde{\psi}$ is explicitly given by

$$\tilde{\psi}((\underbrace{\lambda_i}_{= 0 \text{ for almost all } i})_{i \in I}) = \prod_{i \in I} a_i^{\lambda_i}.$$

Then $\psi := \tilde{\psi} \circ \phi$ is the desired group homomorphism. Indeed, let $w \in \Gamma_{\text{val}}$, i.e. $w = \sum_{i \in I} \lambda_i w_i$ for $\lambda_i \in \mathbb{Q}$ almost all zero. Obtain

$$\operatorname{val}(\psi(w)) = \operatorname{val}(\tilde{\psi} \circ \phi(w)) = \operatorname{val}\Big(\prod_{i \in I} a_i^{\lambda_i}\Big) = \sum_{i \in I} \lambda_i \cdot \underbrace{\operatorname{val}(a_i)}_{=w_i} = w.$$

□

Remark 2.12. The advantage of the proof just given is that it is constructive if the valuation group is finite dimensional as a $\mathbb{Q}$-vector space. By means of homological algebra, however, one can give a much shorter proof for the existence of the splitting: We consider the exact sequence of abelian groups (thus naturally as $\mathbb{Z}$-modules)

$$0 \longrightarrow \ker(\text{val}) \longrightarrow K^* \xrightarrow{\text{val}} \Gamma_{\text{val}} \longrightarrow 0.$$

For an n-th root b of some element $a \in \ker(\text{val})$ we have $\operatorname{val}(b) = \frac{1}{n}\operatorname{val}(a) = 0$, thus $\ker(\text{val})$ is divisible, in particular an injective $\mathbb{Z}$-module and the sequence splits. Hence we get desired homomorphism $\psi \colon \Gamma_{\text{val}} \to K^*$ satisfying $\operatorname{val} \circ \psi = \operatorname{id}_{\Gamma_{\text{val}}}$.

Considering the field of Puiseux series $K\{\{t\}\}$ with its natural valuation, we have $\operatorname{val}(K\{\{t\}\}) = \mathbb{Q}$, and we obtain a splitting $\psi \colon \mathbb{Q} \to K\{\{t\}\},\ w \mapsto t^w$. Reminiscent of this map, if a splitting exists, we will use in general the notation t^w to denote the element $\psi(w) \in K^*$.

Remark 2.13. In 2.11 we needed to assume that the valued field is algebraically closed. In general a splitting does not exist, as we will see now. The following counterexample follows the guided exercise [MS15, Exercise 2.7(6)]:
Consider any field K and let $L := K(X_1, X_2, \dots)$ be the field of rational functions in countably many variables. This can also be expressed as the union $L = \bigcup_{n \in \mathbb{N}_{>0}} K(X_1, \dots, X_n)$. For an $f(X_1, \dots, X_m) = \sum_{i=1}^{k} c_i {X_1}^{u_{i,1}} \cdots {X_m}^{u_{i,m}} \in K[X_1, \dots, X_m]$ with $c_i \in K, u_{i,j} \in \mathbb{N}$, we set

$$v(f) := \min_{i=1,\dots,k} \Big\{ \sum_{j=1}^{m} \frac{u_{i,j}}{j} \Big\}$$

It is easy to check that this yields a valuation $\operatorname{val} \colon L^* \to \mathbb{Q}, \frac{f}{g} \mapsto v(f) - v(g)$ which satisfies $\operatorname{val}(c) = 0$ for all $c \in K$, as well as $\operatorname{val}(X_j) = \frac{1}{j}$ for all $j \in \mathbb{N}_{>0}$, in particular $\Gamma_{\text{val}} = \mathbb{Q}$. We now suppose that a splitting $\phi \colon \mathbb{Q} \to L^*$ exists. Thus there exist $f, g \in K[X_1, \dots, X_m]$ for some $m \in \mathbb{N}$, such that $\phi(1) = \frac{f}{g}$. Note that $f \notin K$, thus not a unit in $K[X_1, \dots, X_m]$. The polynomial ring in finitely many variables is a unique factorization domain. We may assume that f, g have no common prime factors and that f has prime decomposition $f = c \cdot p_1^{l_1} \cdot p_k^{l_k}$, where c is some unit and $k > 0$. Choose $n > \max\{l_1, \dots, l_k\}$, and get $h, i \in K[X_1, \dots, X_{\tilde{m}}]$ for some $\tilde{m} \in \mathbb{N}$, such that $\phi(\frac{1}{n}) = \frac{h}{i}$. Again, we may assume that h, i have no common prime factors, and that $\tilde{m} > m$. The prime factorization of f stays the same in $K[X_1, \dots, X_{\tilde{m}}]$. But then $\frac{h^n}{i^n} = \phi(n \cdot \frac{1}{n}) = \phi(1) = \frac{f}{g}$ is a contradiction to the factorization of f, as $n > \max\{l_1, \dots, l_k\}$.

We close this chapter with an example of how to extend a p-adic valuation on $\mathbb{Q}$ to a number field. Interestingly, this will give as well some incentive for the study of tropical polynomials. It is inspired by [MS15, Example 2.1.16].

Example 2.14. Let K be any number field with ring of integers $\mathcal{O}_K$. Any prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ determines a valuation on K as in Example 2.8. So in order to extend the p-adic valuation for a prime p, one can naturally choose any prime ideal $\mathfrak{p}$ lying above p and define an extending valuation on K by

$$\mathrm{val}_K \colon K^* \to \frac{1}{e_\mathfrak{p}}\mathbb{Z}, \quad x \mapsto \frac{v_\mathfrak{p}(x)}{e_\mathfrak{p}},$$

where $e_\mathfrak{p}$ is the ramification index of $\mathfrak{p}$ over the prime p. The restriction of val_K to $\mathbb{Q}$ coincides with the p-adic valuation. Here we also see that in general an extension of a valuation is not unique.

Now consider for example the imaginary quadratic number field $K := \mathbb{Q}(\sqrt{-29})$ with ring of integers $\mathcal{O}_K$. The ideals $2\mathcal{O}_K$, $3\mathcal{O}_K$, $5\mathcal{O}_K$ factor into a product of prime ideals:

$$2\mathcal{O}_K = \mathfrak{p}^2,\ 3\mathcal{O}_K = \mathfrak{q}_1\mathfrak{q}_2,\ 5\mathcal{O}_K = \mathfrak{r}_1\mathfrak{r}_2,$$

where $\mathfrak{p}, \mathfrak{q}_i, \mathfrak{r}_i$ are distinct prime ideals in $\mathcal{O}_K$. To be precise, have $\mathfrak{p} = (2, 1 + \sqrt{-29})$, $\mathfrak{q}_i = (3, 1 + (-1)^i\sqrt{-29})$ and $\mathfrak{r}_i = (5, 1 + (-1)^i\sqrt{-29})$. The principal ideal $\alpha\mathcal{O}_K$ of the element $\alpha := 1 + \sqrt{-29} \in \mathcal{O}_K$ decomposes into $\alpha\mathcal{O}_K = \mathfrak{p} \cdot \mathfrak{q}_2 \cdot \mathfrak{r}_2$. Thus, for an extension val_K on K of the p-adic valuation we get the following possibilities for $\mathrm{val}_K(\alpha)$:

$$p = 2:\ \ \mathrm{val}_K(\alpha) = \frac{1}{2}$$

$$p = 3, 5:\ \ \mathrm{val}_K(\alpha) = 1 \text{ or } 0$$

We will anticipate some aspects of the following chapters. Given a polynomial $f(X) = \sum_{i=0}^{n} a_i X^i \in F[X]$ with coefficients in some valued field (F, val_F), we define the *tropicalization* of f as the piecewise linear, real function

$$\mathrm{trop}(f)(x) = \min\{\mathrm{val}_F(a_i) + i \cdot x \mid i \in \{0, \dots, n\}\}, \tag{2.5}$$

and *roots* of $\mathrm{trop}(f)$ as the real points x where (2.5) fails to be linear, i.e. where the minimum in $\mathrm{trop}(f)(x)$ is obtained at least twice.

Now consider the minimal polynomial $f(X) = X^2 - 2X + 30 \in \mathbb{Q}[X]$ of α as above. For the p-adic valuation on $\mathbb{Q}$, we get tropical polynomials as follows:

$$\begin{aligned} p = 2: \;\; & \mathrm{trop}(f)(x) = \min\{2x, 1 + x, 1\}, \text{ roots: } \frac{1}{2} \\ p = 3, 5: \;\; & \mathrm{trop}(f)(x) = \min\{2x, x, 1\}, \text{ roots: } 0, 1 \end{aligned}$$

We see that the roots of the tropical polynomials are exactly the possible values of $\mathrm{val}_K(\alpha)$. So algebraic extensions of valued fields seem to be naturally connected to solving tropical polynomial equations.

3

Algebraic Varieties

In this short chapter we take a look at basic concepts of algebraic geometry and start working with the main ring of interest in this thesis: the Laurent polynomial ring in multiple variables.

Definition 3.1. Let K be an algebraically closed field.

i.) The n-dimensional *affine space* $\mathbb{A}^n_K$ over K is

$$\mathbb{A}^n_K := \mathbb{A}^n := \{(a_1, a_2, \dots, a_n) \mid a_i \in K\}.$$

The *coordinate ring* of the affine space $\mathbb{A}^n$ is the polynomial ring $K[X_1, \dots, X_n]$. The n-dimensional *algebraic torus* T^n_K over K is defined as

$$T^n_K := T^n := \{(a_1, a_2, \dots, a_n) \mid a_i \in K^*\}$$

The *coordinate ring* of the algebraic torus T^n is the Laurent polynomial ring $K[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$.

ii.) The *affine variety* $V(I)$ defined by an ideal $I \subseteq K[X_1, \dots, X_n]$ is

$$V(I) := \{a \in \mathbb{A}^n \mid f(a) = 0 \text{ for all } f \in I\}.$$

Similarly we define a *very affine variety* $V(J)$ in the torus for an ideal $J \subseteq K[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$:

$$V(J) := \{a \in T^n \mid f(a) = 0 \text{ for all } f \in J\}.$$

iii.) For a polynomial $f \in K[X_1, \ldots, X_n]$ with

$$f(X_1, \ldots, X_n) = \sum_{(u_1,\ldots,u_n)\in\mathbb{N}^n} c_{(u_1,\ldots,u_n)} X_1^{u_1} \cdots X_n^{u_n},$$

where the coefficients $c_{(u_1,\ldots,u_n)} \in K$ are almost all zero, we often just write in a simpler form $f = \sum_{u\in\mathbb{N}^n} c_u X^u$ and define the *support* of f as $\operatorname{supp}(f) := \{u \in \mathbb{N}^n \mid c_u \neq 0\}$.

The degree of such a polynomial is $\deg(f) = \max_{u\in\mathbb{N}^n} \{|u| \mid c_u \neq 0\}$, where $|u| = \sum_{i=1}^n u_i$. The *homogenization* $\tilde{f}$ of f is the homogenous polynomial $\tilde{f} = \sum_{u\in\mathbb{N}^n} c_u X_0^{\deg(f)-|u|} X^u \in K[X_0, X_1, \ldots, X_n]$. An ideal $I \subseteq K[X_1, \ldots, X_n]$ is *homogenous* if it has a generating set consisting of homogenous polynomials. We similarly define all these terms in the Laurent polynomial ring $K[X_1^{\pm1}, \ldots, X_n^{\pm1}]$.

Remark 3.2. Let K be a field.

i.) For $R := K[X_1, \ldots, X_n]$ we can regard the Laurent polynomial ring as the localization $S^{-1}R$ of R at the multiplicatively closed set $S := \{X_1^{a_1} \cdot X_2^{a_2} \cdots X_n^{a_n} \mid (a_1, \ldots, a_n) \in \mathbb{N}^n\}$. Hence $K[X_1^{\pm1}, \ldots, X_n^{\pm1}]$ inherits several useful properties from R; it is in particular Noetherian.

ii.) Suppose K is algebraically closed. We place the *Zariski topology* on the algebraic torus T^n by taking the closed sets to be

$$\{V(I) \mid I \text{ ideal in } K[X_1^{\pm1}, \ldots, X_n^{\pm1}]\},$$

where $V(I)$ is the very affine variety defined by I. It is easy to see that this defines indeed a topology:

a.) Have $\emptyset = V(\langle 1\rangle)$ and $T^n = V(\langle 0\rangle)$.

b.) The Noetherianity of $K[X_1^{\pm1}, \ldots, X_n^{\pm1}]$ yields that two ideals I, J satisfy $V(I) \cup V(J) = V(I \cdot J) = V(I \cap J)$, as I, J are finitely generated. Indeed, let $I = \langle f_1, \ldots, f_n\rangle$ and $J = \langle g_1, \ldots, g_m\rangle$. Any $a \in V(I \cap J)$ satisfies $f_i g_j(a) = 0$ for all i, j. If there exists g_k with $g_k(a) \neq 0$, then $f_i g_k(a) = 0$ implies $f_i(a) = 0$ for all i, and vice versa. Hence $V(I) \cup V(J) \supseteq V(I \cap J)$. The other inclusion is clear. Thus a finite union of closed sets is closed again.

c.) To see that arbitrary intersections of closed sets are closed again, note that for some index set J and ideals I_j, $j \in J$ we have $\cap_{j\in J} V(I_j) = V\big(\sum_{j\in J} I_j\big)$, where

$$\sum_{j\in J} I_j = \Big\{ \sum_{j\in J} a_j \mid a_j \in I_j, a_j = 0 \text{ for almost all } j \in J \Big\}.$$

Remark 3.3. Let K be field and $T := K^*$, $n, m \in \mathbb{N}_{>0}$. A *monomial map* $T^n \to T^m$ is a map which is specified by m Laurent monomials in $X_1, \ldots, X_n$, i.e. a map

$$\phi\colon T^n \to T^m,\ (y_1, \ldots, y_n) \mapsto (y_1^{a_{11}} \cdots y_n^{a_{n1}}, \ldots, y_1^{a_{1m}} \cdots y_n^{a_{nm}}), \tag{3.1}$$

with $a_{ij} \in \mathbb{Z}$ for $1 \le i \le n, 1 \le j \le m$. We thus see that such a monomial map can also be represented by a matrix $A := (a_{ij})_{ij} \in M(n \times m, \mathbb{Z})$. The m-th column of A defines exactly the exponents of the m-th Laurent monomial. Furthermore we have for $y, z \in T^n$ that $\phi(y \cdot z) = \phi(y) \cdot \phi(z)$ (with componentwise multiplication), so a monomial map defines a group homomorphism between algebraic tori.
Also see easily that for two monomial maps $\phi\colon T^n \to T^m$ represented by $A \in M(n \times m, \mathbb{Z})$ and $\psi\colon T^m \to T^k$ represented by $B \in M(m \times k)$ that the representing matrix of $\psi \circ \phi$ is given by $A \cdot B$.
An *automorphism* of the torus T^n is an invertible monomial map. Hence it follows immediately that the automorphisms of T^n form a group which is canonically isomorphic to $\mathrm{GL}_n(\mathbb{Z})$.
We finally observe that a monomial map as in (3.1) induces a ring homomorphism $\phi^*\colon K[Z_1^{\pm 1}, \ldots, Z_m^{\pm 1}] \to K[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ which is given by $\phi^*(Z_i) = X_1^{a_{1i}} \cdots X_n^{a_{ni}}$. Here a Laurent polynomial $\sum_{u\in\mathbb{Z}^n} c_u X^u$ is mapped to

$$\phi^*\Big(\sum_{u\in\mathbb{Z}^n} c_u Z^u\Big) = \sum_{u\in\mathbb{Z}^n} c_u X^{Au}.$$

If ϕ is an automorphism of T^n, we can also write

$$\phi^*\Big(\sum_{u\in\mathbb{Z}^n} c_u X^u\Big) = \sum_{u\in\mathbb{Z}^n} c_{A^{-1}u} X^u.$$

We conclude this chapter with a lemma which is needed later when discussing tropical hypersurfaces.

Lemma 3.4. Let K be an algebraically closed, nontrivially valued field with a splitting $\Gamma_{\mathrm{val}} \to K^*$, $w \mapsto t^w$ such that $\mathrm{val}(t^w) = w$. Let $\alpha_1, \ldots, \alpha_n \in \Bbbk^*$ and $w_1, \ldots, w_n \in \Gamma_{\mathrm{val}}$, and consider the set

$$U := \{y = (y_1, \ldots, y_n) \in T^n \mid \mathrm{val}(y_i) = w_i,\ \overline{t^{-w_i} y_i} = \alpha_i \text{ for } i = 1, \ldots, n\}$$

Then U is dense in T^n with respect to the Zariski topology.

Proof: Let $B \neq \emptyset$ be open subset of T^n. We need to show $B \cap U \neq \emptyset$. We have $T^n \setminus B = V(I)$ for some nontrivial ideal I. Let $h \in I$ nonzero and $y \in U$ with $h(y) \neq 0$. Then $y \notin V(I) = T^n \setminus B$, so $y \in B$. It thus suffices to show that for any nonzero polynomial $h \in K[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ there is a point $y \in U$ with $h(y) \neq 0$.
Let $h \neq 0$ be a Laurent polynomial. For each i we choose an appropriate lift z_i in the valuation ring R with $\overline{z}_i = \alpha_i$. Then $y_i := t^{w_i} z_i$ satisfies $\mathrm{val}(y_i) = w_i$, as $\alpha_i \neq 0$ implies $\mathrm{val}(z_i) = 0$. Furthermore $\overline{t^{-w_i} y_i} = \overline{t^{-w_i + w_i} z_i} = \alpha_i$.
For each coordinate y_i there is an infinite number of distinct choices in K^*; simply note that Γ_{val} is dense in $\mathbb{R}$ by 2.10 and that $\tilde{y}_i := y_i + t^{w_i + j}$ also satisfies $\mathrm{val}(\tilde{y}_i) = w_i$ and $\overline{t^{-w_i} \tilde{y}_i} = \overline{t^{-w_i} y_i} + \underbrace{\overline{t^j}}_{=0} = \alpha_i$ for all $j > 0$.
We now show by induction on n that there is $y \in U$ with $h(y) \neq 0$. For $n = 1$ we choose y_1 from the infinite number of choices with $\mathrm{val}(y_1) = w_1$ and $\overline{t^{-w_1} y_1} = \alpha_1$ in a way that avoids the finitely many roots of h.
Suppose $n > 1$ and write $h = \sum_{j \in \mathbb{Z}} h_j X_n^j$ for $h_j \in K[X_1^{\pm 1}, \ldots, X_{n-1}^{\pm 1}]$. By induction hypothesis and argumentation with infinite choices as above, there is a $y' = (y_1, \ldots, y_{n-1}) \in (K^*)^{n-1}$ with $\mathrm{val}(y_i) = w_i$ and $\overline{t^{-w_i} y_i} = \alpha_i$ with $h_j(y') \neq 0$ for all j. Then again, we can choose y_n with $\mathrm{val}(y_n) = w_n$ and $\overline{t^{-w_n} y_n} = \alpha_n$ in a way that avoids the finitely many roots of $h(y_1, \ldots, y_{n-1}) \in K[X_n^{\pm 1}]$. $\square$

4

Tropicalization and Initial Forms

In this chapter we introduce tropicalization of Laurent polynomials and of automorphisms of the algebraic torus, as well as initial ideals in a tropical sense. Throughout the chapter, K is always a valued field. We do not need that K is algebraically closed, however we again assume that a splitting $\phi\colon \Gamma_{\mathrm{val}} \to K^*,\ w \mapsto t^w$ exists.

Definition 4.1. i.) Let $f = \sum_{u \in \mathbb{Z}^n} c_u X^u \in K[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$. The *tropicalization* of f is the piecewise linear function $\mathrm{trop}(f)\colon \mathbb{R}^n \to \mathbb{R}$ given by

$$\mathrm{trop}(f)(w) = \min\{\mathrm{val}(c_u) + \langle w, u\rangle \mid u \in \mathbb{Z}^n \text{ and } c_u \neq 0\}. \tag{4.1}$$

Here $\langle u, w\rangle$ denotes the standard Euclidean inner product, i.e. $\langle u, w\rangle = \sum_{i=1}^n u_i w_i$ for $u = (u_1, \ldots, u_n), w = (w_1, \ldots, w_n) \in \mathbb{R}^n$.
Thus $\mathrm{trop}(f)$ is the real valued function on $\mathbb{R}^n$ which is obtained by replacing each coefficient by its valuation and performing all additions and multiplications in the tropical semiring.

ii.) Fix a *weight vector* $w \in \mathbb{R}^n$ and let $W := \mathrm{trop}(f)(w)$. Then the *initial form* of f with respect to w is defined as

$$\mathrm{in}_w(f) := \sum_{\substack{u \in \mathrm{supp}(f),\\ \mathrm{val}(c_u) + \langle u, w\rangle = W}} \overline{t^{-\mathrm{val}(c_u)} c_u} X^u \in \Bbbk[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]. \tag{4.2}$$

Note that if $w \in \Gamma_{\mathrm{val}}^n$ we can also write

$$\mathrm{in}_w(f) = \sum_{u \in \mathbb{Z}^n} \overline{t^{-W + \langle u, w\rangle} c_u} X^u. \tag{4.3}$$

This is indeed well-defined, as in this case $-W + \langle u, w\rangle \in \Gamma_{\mathrm{val}}$.

Example 4.2. i.) Consider the field of Puiseux series $\mathbb{C}\{\{t\}\}$ with its natural valuation and let $f = (t+t^3)X_1 + 3t^4X_2 + 3tX_1X_2 \in \mathbb{C}\{\{t\}\}[X_1, X_2]$. Get $\mathrm{trop}(f)(w) = \min\{1+w_1, 4+w_2, 1+w_1+w_2\}$ for $w = (w_1, w_2) \in \mathbb{R}^2$. Hence for $w = (1,1)$ obtain $\mathrm{in}_w(f) = \overline{t^{-1}(t+t^3)}X_1 = X_1$. For $w = (0,0)$ get $\mathrm{in}_w(f) = X_1 + \overline{3}X_1X_2$.

ii.) The field $K = \mathbb{Q}$ with the p-adic valuation admits a canonical splitting $\mathbb{Z} \to \mathbb{Q}, a \mapsto p^a$. It is easy to see that in the residue field $\Bbbk$ every element in $\mathbb{Z}$ lies in one of the distinct classes of $\overline{0}, \overline{1}, \ldots, \overline{p-1}$. For any $\frac{m}{n} \in \mathbb{Q}$ with $\mathrm{val}(\frac{m}{n}) = 0$ we may assume after appropriate cancelling that $p \nmid n$, so there are $k, l \in \mathbb{Z}$ with $ln + kp = 1$. Thus we get $a := \frac{m}{n} - lm = \frac{m-lmn}{n} = \frac{kpm}{n}$ with $\mathrm{val}(a) > 0$, hence $\overline{\frac{m}{n}} = \overline{lm} \in \Bbbk$ and $\Bbbk$ has exactly p elements. Thus $\Bbbk \cong \mathbb{Z}/p\mathbb{Z}$ and we can regard the initial form of a polynomial as having coefficients in $\mathbb{Z}/p\mathbb{Z}$.

Definition 4.3. Let I be an ideal in $K[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ and $w \in \mathbb{R}^n$. We define the *initial ideal* $\mathrm{in}_w(I)$ as the ideal in $\Bbbk[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ which is generated by the initial forms of elements in I, i.e.

$$\mathrm{in}_w(I) := \langle \mathrm{in}_w(f) \mid f \in I \rangle \subseteq \Bbbk[X_1^{\pm 1}, \ldots, X_n^{\pm 1}].$$

Remark 4.4. Initial forms and ideals play an important role in tropical geometry, as we will see. However, for arbitrary choices of a weight vector w, the initial form $\mathrm{in}_w(f)$ of a Laurent polynomial f may be a unit in $\Bbbk[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ and might hence generate the whole ring. This is exactly the case when the minimum in $\mathrm{trop}(f)(w)$ is achieved only once. If this happens, the initial ideal of any ideal containing f comprises no information at all.
It is one objective of tropical geometry to study the weight vectors $w \in \mathbb{R}^n$ for which the initial ideal is actually a *proper* ideal in $\Bbbk[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$. Kapranov's Theorem, our main result in this thesis, draws an astonishing connection between those special weight vectors and very affine varieties defined by principal ideals.

We will now prove some general facts in order to better understand the structure of initial ideals.
The following lemma shows that the sum of initial forms is again an initial form. The idea of the proof follows [MS15, Lemma 2.4.2], however we need to assume

that the weight vector w lies in Γ_{val}^n, because in general the claim will not be true for any $w \in \mathbb{R}^n$. Consider for example an arbitrarily valued field K with Laurent polynomial ring $K[X^{\pm 1}, Y^{\pm 1}]$ and weight vector $w = (a, b) \in \mathbb{R}^2$, where $a \in \Gamma_{\text{val}}, b \notin \Gamma_{\text{val}}$. Obviously we have $\text{in}_w(X) = X$ and $\text{in}_w(Y) = Y$. Suppose there is $f \in K[X^{\pm 1}, Y^{\pm 1}]$ with $\text{in}_w(f) = X + Y$. Then there exist $c, d \in K^*$ with $\text{val}(c) + a = \text{val}(d) + b$, which implies $b \in \Gamma_{\text{val}}$ in contradiction to our assumption.

Lemma 4.5. Let $w \in \Gamma_{\text{val}}^n$, $f_1, \dots, f_m \in K[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$. Set $W_i := \text{trop}(f_i)(w)$ and suppose $\tilde{g} := \sum_{i=1}^m \text{in}_w(f_i) \neq 0$. Then

$$\sum_{i=1}^m \text{in}_w(f_i) = \text{in}_w\Big(\sum_{i=1}^m t^{-W_i} f_i\Big). \tag{4.4}$$

Proof: Set $g := \sum_{i=1}^m t^{-W_i} f_i$. First note that $W_i \in \Gamma_{\text{val}}$ for all i, so g is actually well-defined. For $i = 1, \dots, m$ write $f_i = \sum_{u \in \mathbb{Z}^n} c_{i,u} X^u$ with $c_{i,u} \in K$. We thus get

$$g = \sum_{u \in \mathbb{Z}^n} a_u X^u, \text{ for } a_u = \sum_{i=1}^m c_{i,u} t^{-W_i}$$

and furthermore

$$\text{trop}(g)(w) = \min_{u \in \text{supp}(g)} \{\text{val}(a_u) + \langle w, u \rangle\} = \min_{u \in \text{supp}(g)} \{\text{val}\Big(\sum_{i=1}^m c_{i,u} t^{-W_i}\Big) + \langle w, u \rangle\}.$$

Have $\text{trop}(g)(w) \geq 0$, as

$$\text{val}(a_u) \geq \min_i \{\text{val}(c_{i,u}) - \text{trop}(f_i)(w)\} \geq -\langle w, u \rangle.$$

Suppose that $\text{trop}(g)(w) > 0$. Choose $u \in \text{supp}(\tilde{g})$ and have $\text{val}(a_u) > -\langle w, u \rangle$. This implies for the u-th coefficient in $\tilde{g}$ that (consider notation of initial form as in (4.3))

$$\text{val}\Big(\sum_{i=1}^m c_{i,u} t^{-W_i + \langle w, u \rangle}\Big) = \text{val}\Big(\big(\sum_{i=1}^m c_{i,u} t^{-W_i}\big) \cdot t^{\langle w, u \rangle}\Big) = \text{val}(a_u) + \langle w, u \rangle > 0,$$

hence $\overline{\sum_{i=1}^m c_{i,u} t^{-W_i + \langle w, u \rangle}} = 0$ in contradiction to $u \in \text{supp}(\tilde{g})$. Note that this argument is indifferent to $a_u = 0$.

Thus $\text{trop}(g)(w) = 0$. We obtain for the u-th coefficient in $\text{in}_w(g)$ (as in (4.3)) for any $u \in \mathbb{Z}^n$ that

$$\overline{a_u t^{-\mathrm{trop}(g)(w)+\langle w,u\rangle}} = \overline{a_u t^{\langle w,u\rangle}} = \overline{\left(\sum_{i=1}^m c_{i,u} t^{-W_i}\right) t^{\langle w,u\rangle}} = \overline{\sum_{i=1}^m c_{i,u} t^{-W_i+\langle w,u\rangle}},$$

which is exactly the u-th coefficient in $\tilde{g}$. This completes the proof. □

The next proposition is a collection of some general results about initial ideals. Note that in the third statement we will again need to assume $w \in \Gamma_{\mathrm{val}}^n$, however the fourth statement holds for all $w \in \mathbb{R}^n$ again. We will see in Remark 4.8 that this immediately implies that a single polynomial already represents a tropical basis for its own principal ideal.

Proposition 4.6. Let $f, g \in K[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ and let $I \subseteq K[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ be an ideal, as well as $w \in \mathbb{R}^n$. Then the following holds:

i.) The initial ideal $\mathrm{in}_w(I)$ admits a finite generating system

$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(f_1), \ldots, \mathrm{in}_w(f_m) \rangle, \; f_1, \ldots, f_m \in I.$$

ii.) Have $\mathrm{in}_w(f \cdot g) = \mathrm{in}_w(f) \cdot \mathrm{in}_w(g)$.

iii.) If $w \in \Gamma_{\mathrm{val}}^n$ and $h \in \mathrm{in}_w(I)$, then $h = \mathrm{in}_w(\tilde{h})$ for some $\tilde{h} \in I$.

iv.) If $X^u \in \mathrm{in}_w(I)$ for some $u \in \mathbb{Z}^n$, then $X^u = \mathrm{in}_w(h)$ for some $h \in I$.

Proof: i.) is clear, because $K[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ is Noetherian by 3.2.
For ii.) consider $f = \sum_{u \in \mathbb{Z}^n} c_u X^u$ and $g = \sum_{u \in \mathbb{Z}^n} d_u X^u \in K[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$, and we may assume that f, g are nontrivial. Get

$$fg = \sum_{v \in \mathbb{Z}^n} e_v X^v, \text{ for } e_v = \sum_{\substack{u, u' \in \mathbb{Z}^n, \\ u+u'=v}} c_u d_{u'}.$$

Let $W_1 := \mathrm{trop}(f)(w), W_2 := \mathrm{trop}(g)(w)$ and we obviously have

$$\mathrm{trop}(fg)(w) = \min_{v \in \mathbb{Z}^n} \{ \underbrace{\mathrm{val}\Big(\sum_{u+u'=v} c_u d_{u'} \Big)}_{\geq \min\{\mathrm{val}(c_u)+\mathrm{val}(d_{u'})\}} + \underbrace{\langle w, v \rangle}_{\substack{=\langle w,u\rangle+\langle w,u'\rangle, \\ \text{for any } u+u'=v}} \} \geq \mathrm{trop}(f)(w) + \mathrm{trop}(g)(w).$$

In order to see equality, consider $\mathbb{Z}^n$ with the lexicographic ordering "$\geq$", i.e. $u \geq u'$ if in the vector difference $u - u'$ the leftmost nonzero entry is ≥ 0. This is a total order, so we choose maximal $z \in \operatorname{supp}(\operatorname{in}_w(f)), z' \in \operatorname{supp}(\operatorname{in}_w(g))$. For $v := z + z'$ obtain

$$\begin{aligned}\operatorname{val}(e_v) + \langle w, v\rangle &= \operatorname{val}\Big(\sum_{\substack{u,u' \in \mathbb{Z}^n,\\ u+u'=v}} c_u d_{u'}\Big) + \langle w, v\rangle \\ &\geq \underbrace{\min_{u+u'=v}\{\operatorname{val}(c_u) + \operatorname{val}(d_{u'})\}}_{\substack{=\operatorname{val}(c_z)+\operatorname{val}(d_{z'}),\\ \text{as } z \in \operatorname{supp}(\operatorname{in}_w(f)), z' \in \operatorname{supp}(\operatorname{in}_w(g))}} + \langle w, v\rangle \end{aligned} \tag{4.5}$$

Let $u, u' \in \mathbb{Z}^n$ with $u + u' = v$ and $\operatorname{val}(c_u) + \operatorname{val}(d_{u'}) = \operatorname{val}(c_z) + \operatorname{val}(d_{z'})$. Then $u \in \operatorname{supp}(\operatorname{in}_w(f)), u' \in \operatorname{supp}(\operatorname{in}_w(g))$. As $u + u' = v = z + z'$, maximality of z, z' yields $0 \geq u - z = z' - u' \geq 0$, thus $z = u$, $z' = u'$. Thus the minimum in (4.5) is obtained only once, which implies equality by Lemma 2.5, and we get

$$\operatorname{val}(e_v) + \langle w, v\rangle = \operatorname{val}(c_z) + \operatorname{val}(d_{z'}) + \langle w, z\rangle + \langle w, z'\rangle = W_1 + W_2.$$

Hence $\operatorname{trop}(fg)(w) = W_1 + W_2$ and we immediately get

$$\begin{aligned}\operatorname{in}_w(fg) &= \sum_{\substack{v \in \mathbb{Z}^n,\\ \operatorname{val}(e_v)+\langle w,v\rangle = W_1+W_2}} \overline{e_v t^{-\operatorname{val}(e_v)}} X^v \\ &= \sum_{\substack{v \in \mathbb{Z}^n,\\ \operatorname{val}(e_v)+\langle w,v\rangle = W_1+W_2}} \sum_{u+u'=v} \overline{c_u d_{u'} t^{-\operatorname{val}(e_v)}} X^v \\ &= \Bigg(\sum_{\substack{u \in \mathbb{Z}^n,\\ \operatorname{val}(c_u)+\langle u,w\rangle = W_1}} \overline{c_u t^{-\operatorname{val}(c_u)}} X^u\Bigg) \cdot \Bigg(\sum_{\substack{u' \in \mathbb{Z}^n,\\ \operatorname{val}(d_{u'})+\langle u',w\rangle = W_2}} \overline{d_{u'} t^{-\operatorname{val}(d_{u'})}} X^{u'}\Bigg) \\ &= \operatorname{in}_w(f) \cdot \operatorname{in}_w(g).\end{aligned}$$

In order to prove iii.), let $w \in \Gamma_{\operatorname{val}}^n$ and nontrivial $h \in \operatorname{in}_w(I)$. By i.) and by considering products of monomials with generators only, we can write $h = \sum_{i=1}^s a_i X^{u_i} \operatorname{in}_w(g_i)$ with $a_i \in \Bbbk^*, u_i \in \mathbb{Z}^n$ for some $s \in \mathbb{N}$ and $g_i \in I$. Choose appropriate lifts c_i in the valuation ring R with $\overline{c_i} = a_i, \operatorname{val}(c_i) = 0$. Then get $\operatorname{in}_w(c_i X^{u_i}) = \overline{c_i t^{-\operatorname{val}(c_i)}} X^{u_i} = a_i X^{u_i}$. With ii.) obtain $h = \sum_{i=1}^s \operatorname{in}_w(\tilde{g}_i)$ for $\tilde{g}_i = c_i X^{u_i} g_i \in I$. The construction in Lemma 4.5 yields an $\tilde{h} \in I$ with $h = \operatorname{in}_w(\tilde{h})$. At last prove iv.). For this let $u \in \mathbb{Z}^n, X^u \in \operatorname{in}_w(I)$. Note that as in iii.), we can write $X^u = \sum_{i=1}^s \operatorname{in}_w(f_i)$ for some $s \in \mathbb{N}$ and $f_i \in I$. If $\operatorname{supp}(\operatorname{in}_w(f_i)) \cap \operatorname{supp}(\operatorname{in}_w(f_j)) \neq \emptyset$ for some $i, j \in \{1, \ldots, s\}$, then $\operatorname{trop}(f_j)(w) - \operatorname{trop}(f_i)(w) = \operatorname{val}(a) - \operatorname{val}(b) \in \Gamma_{\operatorname{val}}$ for some $a, b \in K^*$. A similar argument as in 4.5 shows that

$\mathrm{in}_w(f_i) + \mathrm{in}_w(f_j) = \mathrm{in}_w(g)$ for

$$g = (t^{\mathrm{trop}(f_j)(w) - \mathrm{trop}(f_i)(w)} f_i) + f_j \in I,$$

as g satisfies $\mathrm{trop}(g)(w) = \mathrm{trop}(f_j)(w)$. Hence, after possibly combining them, we can assume that the supports of the $\mathrm{in}_w(f_i)$'s do not intersect. So there can be no coefficient elimination in $X^u = \sum_{i=1}^{s} \mathrm{in}_w(f_i)$, which implies that there is actually only one f_i and this one satisfies $\mathrm{in}_w(f_i) = X^u$. □

We finally show a useful result that initial ideals preserve homogeneity.

Lemma 4.7. Let $I \subseteq K[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ be a homogenous ideal. Fix $w \in \mathbb{R}^n$. Then $\mathrm{in}_w(I)$ is homogenous as well, and there are homogenous polynomials $g_1, \ldots, g_s$ such that $\mathrm{in}_w(I) = \langle \mathrm{in}_w(g_1), \ldots, \mathrm{in}_w(g_s) \rangle$.

Proof: First show that $\mathrm{in}_w(I)$ is homogenous, i.e. that there is a generating system of homogenous polynomials. Let $h_1, \ldots, h_r$ be a finite homogenous generating system of I (finiteness is possible because $K[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ is Noetherian by 3.2), and let $f = \sum_{i=1}^{\tilde{r}} g_i h_i \in I$. If $g_i = \sum_{u \in \mathbb{Z}^n} c_u X^u$ for $i \in \{1, \ldots, \tilde{r}\}$, note that the summands $c_u X^u h_i$ in $g_i h_i = \sum_{u \in \mathbb{Z}^n} c_u X^u h_i$ are all homogenous and are elements in I. Hence we can write $f = \sum_{i=1}^{r} f_i$ for homogenous $f_i \in I$. By possibly combining the f_i's of same degree, we may also assume that $\deg(f_i) \neq \deg(f_j)$ for $i \neq j$. This implies in particular that $\mathrm{supp}(f_i) \cap \mathrm{supp}(f_j) = \emptyset$ for $i \neq j$. Hence we see immediately from the definition that $\mathrm{in}_w(f)$ is the sum of initial forms of those f_i with $\mathrm{trop}(f_i)(w) = \mathrm{trop}(f)(w)$. Since each homogenous component f_i lives in I, the initial ideal $\mathrm{in}_w(I)$ can be generated alone by elements $\mathrm{in}_w(h)$ with $h \in I$ homogenous. By definition, the initial form of a homogenous polynomial is homogenous as well, so $\mathrm{in}_w(I)$ is a homogenous ideal. Then again by Noetherianity, we can reduce this homogenous generating set to a finite generating subset. □

Remark 4.8. It can be shown that every ideal $I \subseteq K[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ possesses a finite generating system $\mathcal{T} = \{f_1, \ldots, f_m \mid f_i \in I\}$ such that for all $w \in \mathbb{R}^n$ the initial ideal $\mathrm{in}_w(I)$ contains a unit if and only if $\mathrm{in}_w(\mathcal{T}) = \{\mathrm{in}_w(f) \mid f \in \mathcal{T}\}$ contains a unit.

Such a generating system of I is called a *tropical basis*. The proof of existence is rather involved; for details see [MS15, Section 2.6]. A tropical basis has the advantage that the question posed in Remark 4.4, if an initial ideal generates the whole ring, can be answered by merely looking at the finitely many elements of its tropical basis. For example, any $f \in K[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ is also a tropical basis of the principal ideal $\langle f \rangle$ it generates. Indeed, suppose that $\mathrm{in}_w(\langle f \rangle)$ contains a monomial h. By Proposition 4.6 iv.), $h = \mathrm{in}_w(fg) = \mathrm{in}_w(f)\mathrm{in}_w(g)$ is a unit for some $g \in K[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$. But then $\mathrm{in}_w(f)$ is of course also a unit.

We close this chapter by constructing tropicalizations of monomial maps of the algebraic torus. After that, we will have gathered all the tools necessary to show Kapranov's Theorem, the main result of this thesis.

Remark 4.9. (Tropicalization of monomial maps) Let K be a valued field with algebraic torus $T := K^*$ and for $n, m \in \mathbb{N}$ consider a monomial map

$$\phi\colon T^n \to T^m, (y_1, \dots, y_n) \mapsto (y_1^{a_{11}} \cdots y_n^{a_{n1}}, y_1^{a_{12}} \cdots y_n^{a_{n2}}, \dots, y_1^{a_{1n}} \cdots y_n^{a_{nm}}),$$

with $a_{ij} \in \mathbb{Z}$. The corresponding matrix to ϕ is thus given by $A = (a_{ij})_{ij} \in M(n \times m, \mathbb{Z})$. Denote by a_k the k-th column of A for $k = 1, \dots, m$. As in 3.3, the map ϕ induces a ring homomorphism $\phi^*\colon K[Z_1^{\pm 1}, \dots, Z_m^{\pm 1}] \to K[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ given by $\phi^*(Z_k) = X^{a_k}$. Also denote by ϕ^* the $\mathbb{Z}$-homomorphism $\phi^*\colon \mathbb{Z}^m \to \mathbb{Z}^n$ given by $\phi^*(e_i) = a_i$, where $e_1, \dots, e_m$ is the standard basis of $\mathbb{Z}^m$. Again, this canonically induces a map $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}^m, \mathbb{Z}), \varphi \mapsto \varphi \circ \phi^*$. We have $\mathbb{Z}^n \cong \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$ by mapping to dual basis, i.e. by sending e_i to the homomorphism $e_i^*\colon \mathbb{Z}^n \to \mathbb{Z}, e_j \mapsto \delta_{ij}$. Here δ_{ij} denotes the Kronecker delta. Putting this together, we get an induced $\mathbb{Z}$-homomorphism, called the *tropicalization* of ϕ:

$$\begin{aligned} \mathrm{trop}(\phi)\colon \mathbb{Z}^n &\to \mathrm{Hom}(\mathbb{Z}^m, \mathbb{Z}) \to \mathbb{Z}^m \\ e_i \mapsto (e_i^* \circ \phi^*\colon e_j \mapsto a_{ij}) &= \sum_{k=1}^{m} a_{ik} e_k^* \mapsto (a_{i1}, \dots, a_{im}). \end{aligned}$$

Hence the corresponding $m \times n$-matrix to $\mathrm{trop}(\phi)$ is exactly $(a_{ji})_{ij} = A^T$. We also denote by $\mathrm{trop}(\phi)$ the vector space homomorphism obtained by passing to tensor product $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}^n$. Recall that we have a canonical isomorphism

$$\mathbb{R}^n \cong \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}^n, (x_1, \dots, x_n) \mapsto \sum_{i=1}^{n} (x_i \otimes e_i),$$

hence a short calculation shows that $\mathrm{trop}(\phi) : \mathbb{R}^n \to \mathbb{R}^m$ is canonically given by $\mathrm{trop}(\phi)(x) = A^T x$.

Note that $\mathrm{trop}(\phi)(\Gamma_{\mathrm{val}}^n) \subseteq \Gamma_{\mathrm{val}}^m$. Indeed, for $y = (y_1, \dots, y_n) \in T^n$ and $\mathrm{val}(y) :=$

$(\mathrm{val}(y_1), \dots, \mathrm{val}(y_n))$ have

$$\begin{aligned} \mathrm{trop}(\phi)(\mathrm{val}(y)) &= A^T \mathrm{val}(y) \\ &= (\langle a_1, \mathrm{val}(y)\rangle, \dots \langle a_m, \mathrm{val}(y)\rangle,) \\ &= (\mathrm{val}(y_1^{a_{11}} \cdots y_n^{a_{n1}}), \dots, \mathrm{val}(y_1^{a_{1m}} \cdots y_n^{a_{nm}})) \\ &= \mathrm{val}(\phi(y)). \end{aligned} \tag{4.6}$$

Given a monomial automorphism ϕ, we will later see that tropicalization of ϕ has the advantage that for a Laurent polynomial f and $w \in \mathbb{R}^n$, the initial form $\mathrm{in}_{\mathrm{trop}(\phi)(f)}(\phi^{-1*}(f))$ inherits many properties from $\mathrm{in}_w(f)$. Hence tropicalization provides a useful tool for studying initial forms under coordinate changes of the torus.

5

Tropical Hypersurfaces

We now introduce the tropical hypersurface. This is the tropicalization of a very affine variety generated by a Laurent polynomial over a valued field.

Definition 5.1. Let K be a field with a (possibly trivial) valuation and $f \in K[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$. The *tropical hypersurface* $\operatorname{trop}(V(f))$ is the set

$$\operatorname{trop}(V(f)) = \{w \in \mathbb{R}^n \mid \text{the minimum in } \operatorname{trop}(f)(w) \text{ is achieved at least twice}\}. \tag{5.1}$$

We see that the tropical hypersurface consists of all points in $\mathbb{R}^n$ where the piecewise linear function $\operatorname{trop}(f)$ fails to be linear. If the valuation on K has a splitting $w \to t^w$, then the notion of initial forms exists and $\operatorname{trop}(V(f))$ can be rephrased as the set of weight vectors $w \in \mathbb{R}^n$ for which the initial form $\operatorname{in}_w(f)$ is not a unit in $\Bbbk[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$. The equivalence of these definitions is a part of Kapranov's Theorem.

Example 5.2. Let $K = \mathbb{C}\{\{t\}\}$ with its natural valuation. Consider the bivariate polynomial $f \in K[X^{\pm 1}, Y^{\pm 1}]$ given by $f = tX^3Y^2 + t^2XY + (1 + t^4)X$. Then $\operatorname{trop}(f)(w_1, w_2) = \min\{1 + 3w_1 + 2w_2, 2 + w_1 + w_2, w_1\}$, and one easily sees that

$$\begin{aligned}\operatorname{trop}(V(f)) = \{w_2 = 1 - 2w_1 \wedge w_1 \geq \frac{3}{2}\} \\ \cup \{w_2 = -\frac{1}{2} - w_1 \wedge w_1 \leq \frac{3}{2}\} \\ \cup \{w_2 = -2 \wedge w_1 \geq \frac{3}{2}\}.\end{aligned}$$

The corresponding graph is the tropical curve shown in Figure 5.1.

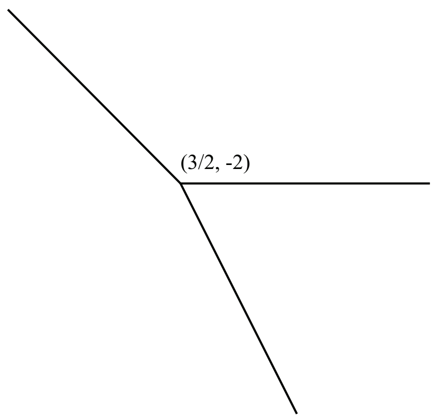

FIGURE 5.1: Tropical curve of $tX^3Y^2 + t^2XY + (1+t^4)X$.

We now formulate Kapranov's Theorem which links the classical hypersurface $V(f)$ with $\mathrm{trop}(V(f))$. However, we need to assume that the field K is algebraically closed and has a nontrivial valuation. For an arbitrary field K with a nontrivial valuation, we can pass to the algebraic closure $\overline{K}$ and extend the valuation to $\overline{K}$, which is possible by Theorem 2.3. If K is trivially valued, we may pass for example to $\overline{K(\!(t)\!)}$ with extension of the usual valuation as in 2.9.

Theorem 5.3. (Kapranov) Let (K, val) be an algebraically closed field with a nontrivial valuation and we fix a Laurent polynomial $f(X) = \sum_{u\in\mathbb{Z}^n} c_u X^u \in K[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$. Then the following three sets conincide:

i.) the tropical hypersurface $\mathrm{trop}(V(f)) \subseteq \mathbb{R}^n$;

ii.) the set $\{w \in \mathbb{R}^n \mid \mathrm{in}_w(f) \text{ is not a monomial}\}$;

iii.) the closure in $\mathbb{R}^n$ of $\{(\mathrm{val}(y_1), \dots, \mathrm{val}(y_n)) \mid (y_1, \dots, y_n) \in V(f)\}$ with respect to the Euclidean topology.

Furthermore, if f is irreducible and $w \in \Gamma_{\mathrm{val}}^n \cap \mathrm{trop}(V(f))$, then the set $\{y \in V(f) \mid \mathrm{val}(y) = w\}$ is Zariski dense in the hypersurface $V(f)$.

Lemma 5.4. Let K be algebraically closed, valued field, and $f = \sum_{u\in\mathbb{Z}^n} c_u X^u \in K[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$. Then the closure $\bar{A}$ with regard to the Euclidean topology of the set $A := \{w \in \Gamma_{\mathrm{val}}^n \mid \mathrm{in}_w(f) \text{ is not a monomial}\}$ is

$$\bar{A} = B := \{w \in \mathbb{R}^n \mid \mathrm{in}_w(f) \text{ is not a monomial}\}.$$

Proof: As K is algebraically closed, the value group Γ_{val} is dense in $\mathbb{R}$ by Lemma 2.10. Thus every open neighborhood of a point in B intersects A, hence we have

$A \subseteq B \subseteq \bar{A}$. So we merely need to show that B is closed, i.e. show that $C := \mathbb{R}^n \setminus B = \{w \in \mathbb{R}^n \mid \mathrm{in}_w(f) \text{ is a monomial}\}$ is open. We give a complete epsilon-delta proof.
Let $w \in C$. Then the minimum in $\mathrm{trop}(f)(w) = \min\limits_{u \in \mathrm{supp}(f)}\{\mathrm{val}(c_u) + \langle u, w\rangle\}$ is achieved only once, i.e. there exists $v \in \mathrm{supp}(f)$ such that for all $u \neq v$ we have $\mathrm{val}(c_v) + \langle v, w\rangle < \mathrm{val}(c_u) + \langle u, w\rangle$.

So given any $u \in \mathrm{supp}(f)$, define the obviously continuous function

$$g_u : \mathbb{R}^n \to \mathbb{R}, x \mapsto \mathrm{val}(c_u) + \langle u, x\rangle.$$

Have $g_v(w) < g_u(w)$ for all $u \in \mathrm{supp}(f) \setminus \{v\}$. Let $\epsilon := \min\limits_{u \in \mathrm{supp}(f)\setminus\{v\}}\{g_u(w) - g_v(w)\} > 0$. Because of continuity of the g_u's, for each $u \in \mathrm{supp}(f) \setminus \{v\}$ get $\delta_u > 0$ such that $|g_u(w) - g_u(\tilde{w})| < \frac{\epsilon}{2}$ for all $\tilde{w}$ in the open ball $B_{\delta_u}(w)$ around w. Set $\delta := \min\limits_{u \in \mathrm{supp}(f)\setminus\{v\}}\{\delta_u\}$ and let $x \in B_\delta(w)$.

Then for all $u \in \mathrm{supp}(f) \setminus \{v\}$ we have

$$\underbrace{g_u(x)}_{>g_u(w)-\frac{\epsilon}{2}} - \underbrace{g_v(x)}_{<g_v(w)+\frac{\epsilon}{2}} > g_u(w) - \frac{\epsilon}{2} - g_v(w) - \frac{\epsilon}{2} \underset{g_u(w)-g_v(w)\geq\epsilon}{\geq} \epsilon - \epsilon = 0.$$

Thus the minimum in $\mathrm{trop}(f)(x)$ is also achieved only once and $\mathrm{in}_x(f)$ is a monomial, which implies $B_\delta(w) \subseteq C$ and C is open. $\square$

Proof of Theorem 5.3: Let $w = (w_1, \ldots, w_n) \in \mathrm{trop}(V(f))$. By definition (5.1), the minimum in $W := \mathrm{trop}(f)(w)$ is achieved at least twice. Therefore, by definition of initial forms, $\mathrm{in}_w(f)$ is not a monomial. Thus set i.) is contained in set ii.). Conversely the same: if $\mathrm{in}_w(f)$ is not a monomial, the minimum in W is achieved at least twice, so $w \in \mathrm{trop}(V(f))$. So set i.) and set ii.) are equal.
We now prove that set iii.) is contained in set i.). By Lemma 5.4 set ii.) (and thus set i.)) is closed, so it is enough to show that points of the form $\mathrm{val}(y) = (\mathrm{val}(y_1), \ldots, \mathrm{val}(y_n))$ where $y = (y_1, \ldots, y_n) \in V(f)$ lie in set i.).
Let $y \in V(f)$, then $f(y) = \sum_{u \in \mathbb{Z}^n} c_u y^u = 0$. This implies

$$\mathrm{val}\Big(\sum_{u \in \mathbb{Z}^n} c_u y^u\Big) = \mathrm{val}(0) = \infty > \mathrm{val}(c_v y^v)$$

for all $v \in \operatorname{supp}(f)$. By Lemma 2.5, this means that the minimum in

$$\min_{v \in \operatorname{supp}(f)} \{\operatorname{val}(c_v y^v)\} = \min_{v \in \operatorname{supp}(f)} \{\operatorname{val}(c_v) + \langle v, \operatorname{val}(y)\rangle\}$$

is achieved at least twice. Hence $\operatorname{val}(y) \in \operatorname{trop}(V(f))$ and set iii.) is subset of set i.).

It remains to show that set i.) is contained in iii.). This result follows directly from the following proposition and Lemma 5.4. Proposition 5.5 states that a root of an initial form lifts to a root of its defining Laurent polynomial. Note here that by 2.7 the residue field $\Bbbk$ is algebraically closed as well. So assuming that a root of an initial form exists does not pose any restriction. In addition, the proposition also shows that the set $\{y \in V(f) \mid \operatorname{val}(y) = w\}$ is Zariski dense when f is irreducible, and we are done. □

Proposition 5.5. Let K be algebraically closed, valued field and $f = \sum_{u \in \mathbb{Z}^n} c_u X^u \in K[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$. Let $w = (w_1, \ldots, w_n) \in \Gamma_{\mathrm{val}}^n$. Suppose that $\operatorname{in}_w(f)$ is not a monomial and $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\Bbbk^*)^n$ satisfies $\operatorname{in}_w(f)(\alpha) = 0$.

Then there exists $y = (y_1, \ldots, y_n) \in T^n$ satisfying $f(y) = 0$, $\operatorname{val}(y) = w$, and $\overline{t^{-w_i} y_i} = \alpha_i$ for $1 \leq i \leq n$.

Furthermore, if f is irreducible, then the set of such y is Zariski dense in the hypersurface $V(f)$.

We will use in the proof of 5.5 that these properties of f are invariant under coordinate change, which is the statement of the following lemma. It is a generalization of the special case used in the proof of [MS15, Proposition 3.1.5].

Lemma 5.6. Let f, w, α and y as in 5.5.

Let ϕ be a monomial automorphism on T^n. Also denote by ϕ the monomial automorphism induced on $(\Bbbk^*)^n$. Set $\tilde{\alpha} := \phi(\alpha), \tilde{w} := \operatorname{trop}(\phi)(w), \tilde{f} := \phi^{-1*}(f)$ and $\tilde{y} := \phi(y)$. Then $\tilde{f}, \tilde{w}, \tilde{\alpha}$ and $\tilde{y}$ satisfy the same properties as in 5.5, i.e.:

i.) $\tilde{w} \in \Gamma_{\mathrm{val}}^n$;

ii.) $\operatorname{in}_{\tilde{w}}(\tilde{f})$ is not a monomial;

iii.) $\operatorname{in}_{\tilde{w}}(\tilde{f})(\tilde{\alpha}) = 0$;

iv.) $\tilde{f}(\tilde{y}) = 0$;

v.) $\operatorname{val}(\tilde{y}) = \tilde{w}$;

vi.) $\overline{t^{-\tilde{w}_i}\tilde{y}_i} = \tilde{\alpha}_i$.

Proof: Let $A \in GL_n(\mathbb{Z})$ be the matrix representing the monomial map as in Remark 3.3 and have $\tilde{f} = \sum_{u\in\mathbb{Z}^n} c_u X^{A^{-1}u} = \sum_{u\in\mathbb{Z}^n} c_{Au}X^u$. By (4.6), we have $\mathrm{val}(\tilde{y}) = \mathrm{trop}(\phi)(\mathrm{val}(y)) = \mathrm{trop}(\phi)(w) = \tilde{w}$, which shows i.) and v.). The rest follows easily by simply plugging in the identities. We show ii.) and iii.) exemplarily.
For ii.) note that $\mathrm{supp}(\tilde{f}) = A^{-1}\cdot \mathrm{supp}(f)$ and

$$\begin{aligned}\mathrm{trop}(\tilde{f})(\tilde{w}) = \mathrm{trop}(\phi^{-1*}(f))(\mathrm{trop}(\phi)(w)) &= \min_{u\in\mathrm{supp}(\tilde{f})}\{\mathrm{val}(c_{Au}) + \langle u, A^T w\rangle\}\\ &= \min_{u\in\mathrm{supp}(f)}\{\mathrm{val}(c_{A^{-1}Au}) + \langle A^{-1}u, A^T w\rangle\}\\ &= \min_{u\in\mathrm{supp}(f)}\{\mathrm{val}(c_u) + \langle u, w\rangle\}\\ &= \mathrm{trop}(f)(w).\end{aligned} \tag{5.2}$$

In particular in (5.2) we see that the minimum in $\mathrm{trop}(\tilde{f})(\tilde{w})$ is achieved only once if and only if it is achieved only once in $\mathrm{trop}(f)(w)$, which shows ii.).
For iii.) set $W := \mathrm{trop}(f)(w)$. With (5.2) obtain

$$\begin{aligned}\mathrm{in}_{\tilde{w}}(\tilde{f})(\tilde{\alpha}) &= \sum_{\substack{u\in\mathrm{supp}(\tilde{f}),\\ \mathrm{val}(c_{Au})+\langle u,A^T w\rangle = W}} \overline{t^{-\mathrm{val}(c_{Au})}c_{Au}}\phi(\alpha)^u\\ &= \sum_{\substack{u\in\mathrm{supp}(f),\\ \mathrm{val}(c_u)+\langle u,w\rangle = W}} \overline{t^{-\mathrm{val}(c_u)}c_u}\alpha^{AA^{-1}u}\\ &= \mathrm{in}_w(f)(\alpha) = 0.\end{aligned}$$

□

Proof of Proposition 5.5: Show the claim by induction on n. Let $n = 1$. As we are looking for a root of f, and as forming of inital forms commutes with multiplication by units by 4.6, we may multiply f by a unit if necessary and assume that $f(X) = \sum_{i=0}^s c_i X^i$ where $c_0, c_s \neq 0$. As K is algebraically closed, get $f = \prod_{i=1}^s (a_i X - b_i)$ for appropriate $a_i, b_i \in K^*$. Hence $\mathrm{in}_w(f) = \sum_{i=1}^s \mathrm{in}_w(a_i X - b_i)$ by Proposition 4.6 ii.). So there exists some $0 \leq j \leq s$ such that $\mathrm{in}_w(a_j X - b_j)(\alpha) = 0$. As $\alpha \in \Bbbk^*$, this implies that $\mathrm{in}_w(a_j X - b_j)$ is not a monomial. Hence by definition of initial forms we must have $\mathrm{val}(a_j) + w = \mathrm{val}(b_j)$, and

$\mathrm{in}_w(a_jX - b_j) = \overline{t^{-\mathrm{val}(b_j)+w}a_j}X - \overline{t^{-\mathrm{val}(b_j)}b_j}$, so $\alpha = \overline{t^{-w}b_j/a_j}$. Set $y := b_j/a_j \in K^*$. Then obviously $f(y) = 0$, $\mathrm{val}(y) = \mathrm{val}(b_j) - \mathrm{val}(a_j) = w$, and $\overline{t^{-\mathrm{val}(y)}y} = \alpha$ as required.

We now assume $n > 1$ and that the claim holds for smaller dimensions. We want to reduce now to the case where no two monomials appearing in f carry the same power of X_n. This has the consequence that, when f is regarded as a polynomial in X_n with coefficients in $K[X_1^{\pm 1}, \dots, X_{n-1}^{\pm 1}]$, the coefficients are all monomials of the form d_uX^u for $d_u \in K$ and $u \in \mathbb{Z}^{n-1}$.

In order to show this, let $l \in \mathbb{N}_{>0}$ and consider the monomial map ϕ_l, where the j-th column of the representation matrix A_l is given by $(\delta_{1j}, \dots, \delta_{n-1,j}, l^j)$ for $1 \le j \le n-1$ and $(0, \dots, 0, 1)$ for $j = n$. Here δ_{ij} again denotes the Kronecker delta. It is clear that ϕ_l is an automorphism, as A_l is obviously invertible. The induced automorphism ϕ_l^* on the Laurent polynomial ring is given by $\phi_l^*(X_j) = X_jX_n^{l^j}$ for $1 \le j \le n-1$ and $\phi_l^*(X_n) = X_n$. Now given any $u = (u_1, \dots, u_{n-1}) \in \mathbb{Z}^{n-1}$ we have

$$\phi_l^*(X^uX_n^i) = X^uX_n^{i+\sum_{j=1}^{n-1} u_jl^j}.$$

If we choose l very large, we obtain a polynomial $\phi_l^*(f)$ where each monomial carries a different power of X_n as desired. So by Lemma 5.6, we can reduce the claim to those special $\phi_l^*(f)$'s (once an appropriate y has been found, simply apply Lemma 5.6 with $\phi_l^*(f)$, $\mathrm{trop}(\phi_l^{-1})(w)$, $\phi_l^{-1}(\alpha)$), $\phi_l^{-1}(y)$ and monomial automorphism ϕ_l).

So we now assume that f has this special form as above. We consider the set of all $y := (y_1, \dots, y_{n-1}) \in T^{n-1}$ with $\mathrm{val}(y_i) = w_i$ and $\overline{t^{-w_i}y_i} = \alpha_i$ for $1 \le i \le n-1$. By 3.4 this set is Zariski dense in T^{n-1}. Because of f's special form, for all such choices the polynomial $g(X_n) := f(y_1, \dots, y_{n-1}, X_n)$ is not zero.

We write u' for the projection of a vector $u \in \mathbb{Z}^n$ onto its first $n-1$ coordinates. Also by f's special form, writing $g(X_n) = \sum_{i \in \mathbb{Z}} d_iX_n^i$, we have for all $d_i \neq 0$ the coefficient $d_i = c_uy^{u'}$ for a unique $u = (u_1, \dots, u_n) \in \mathbb{Z}^n$ which has $u_n = i$. Furthermore note that

$$\mathrm{val}(d_i)+w_ni = \mathrm{val}(c_u)+\mathrm{val}(y^{u'})+w_ni = \mathrm{val}(c_u)+\langle w', u'\rangle+w_nu_n = \mathrm{val}(c_u)+\langle w, u\rangle,$$

therefore $\mathrm{trop}(g)(w_n) = \mathrm{trop}(f)(w)$, and

$$\begin{aligned}
\mathrm{in}_{w_n}(g) &= \sum_{\substack{i\in\mathrm{supp}(g),\\ \mathrm{val}(d_i)+w_n i=\mathrm{trop}(g)(w_n)}} \overline{t^{-\mathrm{val}(d_i)}d_i} X_n^i \\
&\underset{(1)}{=} \sum_{\substack{u\in\mathrm{supp}(f),\\ \mathrm{val}(c_u y^{u'})+w_n u_n=\mathrm{trop}(g)(w_n)}} \overline{t^{-\mathrm{val}(c_u)}c_u t^{-\langle u',w'\rangle} y^{u'}} X_n^{u_n} \\
&\underset{(2)}{=} \sum_{\substack{u\in\mathrm{supp}(f),\\ \mathrm{val}(c_u)+\langle w,u\rangle=\mathrm{trop}(f)(w)}} \overline{t^{-\mathrm{val}(c_u)}c_u} \cdot \alpha_1^{u_1}\cdots\alpha_{n-1}^{u_{n-1}} X_n^{u_n} \\
&= \mathrm{in}_w(f)(\alpha_1,\dots,\alpha_{n-1},X_n).
\end{aligned}$$

For equality (1) note that $\mathrm{val}(y^{u'}) = \langle u', w'\rangle$ and (2) follows from $\overline{t^{-w_i}y_i} = \alpha_i$ for $1 \le i \le n-1$.

Hence $\mathrm{in}_{w_n}(g)(\alpha_n) = 0$. By the $n=1$ case there is $y_n \in K^*$ with $\mathrm{val}(y_n) = w_n$ and $\overline{t^{-w_n}y_n} = \alpha_n$ for which $g(y_n) = 0$, and thus $f(y_1,\dots,y_{n-1},y_n) = 0$. We conclude $(y_1,\dots,y_n)$ is the required point in the hypersurface $V(f)$.

It remains to show that if f is irreducible, then the set $\mathcal{Y} = \{y = (y_1,\dots,y_n) \in V(f) \mid \mathrm{val}(y_i) = w_i, \overline{t^{-w_i}y_i} = \alpha_i \text{ for } 1 \le i \le n\}$ is Zariski dense in $V(f)$. So we need to show that any nonempty open set in $V(f)$ intersects $\mathcal{Y}$. Let $U \subseteq V(f)$ be nonempty open set in $V(f)$. Then $V(f) \setminus U = V(I) \cap V(f)$ for some ideal $I \subseteq K[X_1^{\pm 1},\dots,X_n^{\pm 1}]$. We need to find a point $y \in \mathcal{Y}$ with $y \notin V(I)$. Suppose we have $\mathcal{Y} \subseteq V(I)$, in particular any $g \in I$ satisfies $\mathcal{Y} \subseteq V(g)$. If we can show that $g \in \langle f\rangle$, then already $I \subseteq \langle f\rangle$ and $V(f) \subseteq V(I)$, hence $V(I) \cap V(f) = V(f)$ and $U = \emptyset$ in contradiction to our assumption.

So it suffices to show that any $g \in K[X_1^{\pm 1},\dots,X_n^{\pm 1}]$ with $\mathcal{Y} \subseteq V(g)$ satisfies $g \in \langle f\rangle$.

Given any $(y_1,\dots,y_{n-1}) \in T^{n-1}$, with $\mathrm{val}(y_i) = w_i$ and $\overline{t^{-w_i}y_i} = \alpha_i$ for all i, we have just shown how to construct a point $y = (y_1,\dots,y_{n-1},y_n) \in \mathcal{Y}$. The set of such $(y_1,\dots,y_{n-1})$ is Zariski dense in T^{n-1} by Lemma 3.4. This means that the projection of $\mathcal{Y}$ onto the first $n-1$ coordinates cannot be contained in $V(h)$ for any nontrivial $h \in K[X_1^{\pm 1},\dots,X_{n-1}^{\pm 1}]$. Now suppose there exists a nonzero $g \in K[X_1^{\pm 1},\dots,X_n^{\pm 1}]$ with $\mathcal{Y} \subseteq V(g)$. We have to show that g is a multiple of f. Have $\langle f, g\rangle \cap K[X_1^{\pm 1},\dots,X_{n-1}^{\pm 1}] = \{0\}$ (otherwise $\mathcal{Y}$ would be contained in a hypersurface in T^{n-1}). We claim, that since f is irreducible, this already implies that $g \in \langle f\rangle$. This follows from a series of basic algebraic arguments which we will nevertheless write out in detail:

Denote $A := K[X_1^{\pm 1}, \ldots, X_{n-1}^{\pm 1}]$ (so $K[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] = A[X_n^{\pm 1}]$). After multiplying by appropriate powers of X_n, we may get $\tilde{f}, \tilde{g} \in A[X_n] =: A[X]$ with nonzero constant term. As $\langle f, g \rangle \cap K[X_1^{\pm 1}, \ldots, X_{n-1}^{\pm 1}] = \{0\}$, we must have $\deg(\tilde{f}), \deg(\tilde{g}) \geq 1$ in $A[X]$. This implies that $\tilde{f}$ is irreducible in $A[X]$. Indeed, suppose $\tilde{f} = f \cdot X_n^c$ for some $c \in \mathbb{Z}$ and suppose $\tilde{f}$ is reducible, so $\tilde{f} = ab$ for non-units $a, b \in A[X]$, in particular a, b are not monomials of $A = K[X_1^{\pm 1}, \ldots, X_{n-1}^{\pm 1}]$. As the constant term of $\tilde{f}$ is not zero, both a, b also cannot be monomials in $K[X_1^{\pm 1}, \ldots, X_{n-1}^{\pm 1}, X_n]$. Have $f = abX_n^{-c}$. As f is irreducible, this would however imply that a or b is a unit in $A[X_n^{\pm 1}]$, hence monomial in $K[X_1^{\pm 1}, \ldots, X_{n-1}^{\pm 1}, X_n]$, which is a contradiction.
The Laurent polynomial ring A is a unique factorization domain as it is a localization of a polynomial ring in multiple variables.
Now we know that $\tilde{f}, \tilde{g} \notin A$ by assumption and that $\tilde{f}$ is non-constant, irreducible in $A[X]$ and hence also primitive. It is obvious that if $\tilde{f} \mid \tilde{g}$ in $A[X]$, then also $f \mid g$ in $A[X_n^{\pm 1}]$. So assume $\tilde{f} \nmid \tilde{g}$ in $A[X]$. Let Q be the quotient field of the unique factorization domain A. As $\tilde{f}$ primitive, $\tilde{f} \nmid \tilde{g}$ in $Q[X]$. By Gauss Lemma, $\tilde{f}$ stays irreducible in $Q[X]$. But as $Q[X]$ is a principal ideal domain, $\langle \tilde{f} \rangle$ is maximal and thus $\langle \tilde{f}, \tilde{g} \rangle = Q[X]$, so there exist $a, b \in Q[X]$ with $a\tilde{f} + b\tilde{g} = 1$. Clearing the denominators of the coefficients yields a nonzero element in $\langle \tilde{f}, \tilde{g} \rangle \cap A \implies \langle f, g \rangle \cap A \neq \{0\}$ in $A[X_n^{\pm 1}] = K[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$, which is a contradiction.
This completes the proof. □

We finally close this chapter with a short example confirming Kapranov's Theorem.

Example 5.7. Once more let $K = \mathbb{C}\{\{t\}\}$ with the usual valuation. Consider the bivariate polynomial $f = XY + Y + 1 \in K[X^{\pm 1}, Y^{\pm 1}]$ with $\operatorname{trop}(f)(w_1, w_2) = \min\{w_1 + w_2, w_2, 0\}$ and

$$\operatorname{trop}(V(f)) = \{w_1 = 0 \wedge w_2 \leq 0\} \cup \{w_2 = -w_1 \wedge w_1 \leq 0\} \cup \{w_2 = 0 \wedge w_1 \geq 0\}.$$

The curve is depicted in Figure 5.2.

Furthermore get

$$V(f) = \{\Big(z, -\frac{1}{z+1}\Big) \mid z \in K^* \setminus \{-1\}\}.$$

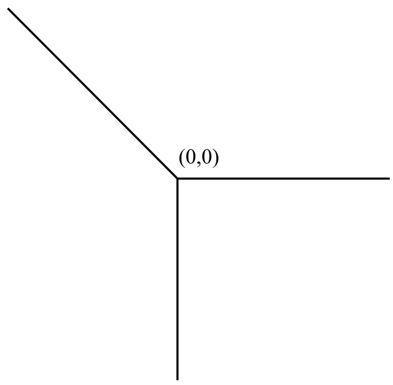

FIGURE 5.2: Tropical curve of $XY + Y + 1$.

From $\mathrm{val}(-\frac{1}{z+1}) = -\mathrm{val}(z+1) \leq -\min\{\mathrm{val}(z), 0\}$, where equality holds if $\mathrm{val}(z) \neq 0$, we obtain

$$(\mathrm{val}(z), \mathrm{val}(-\frac{1}{z+1})) = \begin{cases} (\mathrm{val}(z), 0) & \text{if } \mathrm{val}(z) > 0, \\ (\mathrm{val}(z), -\mathrm{val}(z)) & \text{if } \mathrm{val}(z) < 0, \\ (0, -\mathrm{val}(z+1)) & \text{if } \mathrm{val}(z+1) > 0, \\ (0, 0) & \text{otherwise.} \end{cases} \tag{5.3}$$

As z runs over $K^* \setminus \{-1\}$, the case distinction (5.3) describes all points in Γ_{val}^2 which lie in $\mathrm{trop}(V(f))$. We recall that since K is algebraically closed, Γ_{val} is dense in $\mathbb{R}$ by Lemma 2.10, so the closure of these points is indeed the entire tropical curve as claimed.

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