Forms on the Analytification of Algebraic Varieties

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Lecture in the Seminar on Non-Archimedean Analytic Geometry
December 4th 2018

1. INTRODUCTION

1.1. Motivated by (p,q)-forms in complex analytic geometry, we will define in this lecture the space $A^{p,q}(U)$ of superforms of bidegree (p,q) on an open subset $U \subseteq \mathbb{R}^r$. These superforms can be restricted to the support of polyhedral complexes in $\mathbb{R}^r$. By the Bieri-Groves-Theorem, a closed subscheme of the split torus $\mathrm{Spec}(K[T_1^{\pm 1}, \ldots, T_r^{\pm 1}])$ maps to the support of a polyhedral complex in $\mathbb{R}^r$. We want to use this connection to define differential forms on the analytification X^{an} of an algebraic variety X over some algebraically closed field K. For this we introduce the notion of *very affine open subsets*, i.e. open affine subsets of X which embed as closed subschemes into some split torus. These form a basis for the Zariski topology on X. We will then see that any open subset V of X^{an} can be covered by open subsets of very affine open subsets which behave well with respect to the tropical coordinates. This notion then allows us to define differential forms on X^{an}, with an associated sheaf.

1.2. Let N be a free abelian group of rank r with dual abelian group $M := \mathrm{Hom}(N, \mathbb{Z})$ and associated real vector spaces $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ respectively $M_{\mathbb{R}}$ of dimension r. The choice of a $\mathbb{Z}$-basis of N induces isomorphisms $N \cong \mathbb{Z}^r, N_{\mathbb{R}} \cong \mathbb{R}^r, M_{\mathbb{R}} \cong \mathbb{R}^{r*}$ and leads to coordinates $x_1, \ldots, x_r$ on $N_{\mathbb{R}}$. Our following constructions will only depend on the underlying integral $\mathbb{R}$-affine structures and not on the choice of coordinates. Here an integral $\mathbb{R}$-affine space is a real affine space whose underlying vector space comes with a lattice. Hence we restrict ourselves to the case $N = \mathbb{Z}^r$ with standard basis $e_1, \ldots, e_r$. Note that in subsequent sections in the general case the algebraic torus $\mathrm{Spec}(K[N])$ with character group N takes the role of the torus $\mathrm{Spec}(K[T_1^{\pm 1}, \ldots, T_r^{\pm 1}])$ of rank r.

2. SUPERFORMS ON $\mathbb{R}^r$

2.1. **Definition.**

i.) For an open subset $U \subseteq \mathbb{R}^r$ we denote by $A^p(U)$ the space of smooth real differential forms of degree p. We define the space of *superforms of bidegree* (p,q) on U as

$$A^{p,q}(U) := A^p(U) \otimes_{\mathcal{C}^\infty(U)} A^q(U) = A^p(U) \otimes_{\mathbb{R}} \Lambda^q \mathbb{R}^{r*} = \mathcal{C}^\infty(U) \otimes_{\mathbb{R}} \Lambda^p \mathbb{R}^{r*} \otimes_{\mathbb{R}} \Lambda^q \mathbb{R}^{r*}.$$

ii.) With choice of a basis $x_1, \dots, x_r$ of $\mathbb{R}^r$ we can formally write a superform $\alpha \in A^{p,q}(U)$ as

$$\alpha = \sum_{|I|=p,|J|=q} \alpha_{IJ} d'x_I \wedge d''x_J,$$

where $I = \{i_1, \dots, i_p\}$ respectively $J = \{j_1, \dots, j_q\}$ are ordered subsets of $\{1, \dots, r\}$, $\alpha_{IJ} \in \mathcal{C}^\infty(U)$ are smooth functions and

$$d'x_I \wedge d''x_J := (dx_{i_1} \wedge \dots \wedge dx_{i_p}) \otimes_{\mathbb{R}} (dx_{j_1} \wedge \dots \wedge dx_{j_q}).$$

iii.) We define the wedge product

$$\begin{aligned} A^{p,q}(U) \times A^{p',q'}(U) &\to A^{p+p',q+q'}(U) \\ (\alpha, \beta) &\mapsto \alpha \wedge \beta \end{aligned}$$

in coordinates as

$$\begin{aligned} \alpha \wedge \beta &:= \left(\sum_{|I|=p,|J|=q} \alpha_{IJ} d'x_I \wedge d''x_J \right) \wedge \left(\sum_{|K|=p',|L|=q'} \beta_{KL} d'x_K \wedge d''x_L \right) \\ &:= (-1)^{p'q} \sum_{|I|=p,|J|=q,|K|=p',|L|=q'} \alpha_{IJ}\beta_{KL} d'x_I \wedge d'x_K \wedge d''x_J \wedge d''x_L, \end{aligned}$$

where $d'x_I \wedge d'x_K \in \Lambda^{p+p'}\mathbb{R}^{r*}$ respectively $d''x_J \wedge d''x_L \in \Lambda^{q+q'}\mathbb{R}^{r*}$ is the usual wedge product.

iv.) There is a *differential operator*

$$d' \colon A^{p,q}(U) = A^p(U) \otimes_{\mathbb{R}} \Lambda^q\mathbb{R}^{r*} \to A^{p+1} \otimes_{\mathbb{R}} \Lambda^q\mathbb{R}^{r*} = A^{p+1,q}(U)$$

given by $D \otimes_{\mathbb{R}} \mathrm{id}$ where D is the usual exterior derivative on $A^p(U)$. Also note that $A^{p,q} = \Lambda^p\mathbb{R}^{r*} \otimes_{\mathbb{R}} A^q(U)$, and we define a second operator $d'' := (-1)^p \cdot \mathrm{id} \otimes_{\mathbb{R}} D$. In coordinates this gives

$$d' \left(\sum_{|I|=p,|J|=q} \alpha_{IJ} d'x_I \wedge d''x_J \right) = \sum_{|I|=p,|J|=q} \sum_{i=1}^{r} \frac{\partial \alpha_{IJ}}{\partial x_i} d'x_i \wedge d'x_I \wedge d''x_J$$

and

$$d'' \left(\sum_{|I|=p,|J|=q} \alpha_{IJ} d'x_I \wedge d''x_J \right) = (-1)^p \sum_{|I|=p,|J|=q} \sum_{i=1}^{r} \frac{\partial \alpha_{IJ}}{\partial x_i} d'x_I \wedge d''x_i \wedge d''x_J.$$

Finally define $d := d' + d''$.

2.2. **Remark.** As in differential geometry, we may view a superform

$$\alpha = \sum_{i=1}^{n} \alpha_i \otimes \omega_i \otimes \mu_i \in A^{p,q}(U) = \mathcal{C}^\infty(U) \otimes \Lambda^p \mathbb{R}^{r*} \otimes \Lambda^q \mathbb{R}^{r*}$$

at a point $x \in U$ as a multilinear map

$$\mathbb{R}^{p+q} \to \mathbb{R}, (n_1, \ldots, n_{p+q}) \mapsto \sum_{i=1}^{n} \alpha_i(x)\omega_i(n_1, \ldots, n_p)\mu_i(n_{p+1}, \ldots, n_{p+q})$$

which is alternating in $(n_1, \ldots, n_p)$ and $(n_{p+1}, \ldots, n_{p+q})$. We write $\langle \alpha(x); n_1, \ldots, n_{p+q} \rangle$ for a superform α and such an evaluation at $x \in U$ and $n_1, \ldots, n_{p+q} \in \mathbb{R}^r$.

2.3. **Remark.**

i.) For superforms α respectively β of degree (p, q) respectively (p', q') one computes easily the relations

$$d'(\alpha \wedge \beta) = d'\alpha \wedge \beta + (-1)^{p+q} \alpha \wedge d'\beta$$

and similarly

$$d''(\alpha \wedge \beta) = d''\alpha \wedge \beta + (-1)^{p+q} \alpha \wedge d''\beta.$$

Hence the choice of sign in d''.

ii.) Note that we have as usual $d'(d'\alpha) = 0$ and $d''(d''\alpha) = 0$, however in general not $d'(d''\alpha) = 0$. Indeed, for $\mathbb{R}^2$ with coordinates x, y consider the superform $xy \in A^{0,0}(\mathbb{R}^2) = \mathcal{C}^\infty(\mathbb{R}^2)$. Then $d'(d''(xy)) = d'(yd''x + xd''y) = d'y \wedge d''x + d'x \wedge d''y \neq 0$.

2.4. **Remark.**

i.) Let $F\colon \mathbb{R}^{r'} \to \mathbb{R}^r$ with $F(x) = f(x) + a$ be an affine map. Here f is the corresponding linear map and $a \in \mathbb{R}^r$. Furthermore let $U' \subseteq \mathbb{R}^{r'}$ and $U \subseteq \mathbb{R}^r$ with $F(U') \subseteq U$.

Note that f induces a map $f^*\colon \mathbb{R}^{r*} \to \mathbb{R}^{r'*}$ which again induces a map $f^*\colon \Lambda^k \mathbb{R}^{r*} \to \Lambda^k \mathbb{R}^{r'*}$. In particular we obtain a well-defined *pullback morphism*

$$F^*\colon A^{p,q}(U) = \mathcal{C}^\infty(U) \otimes_{\mathbb{R}} \Lambda^p \mathbb{R}^{r*} \otimes_{\mathbb{R}} \Lambda^q \mathbb{R}^{r*} \to A^{p,q}(U')$$
$$g \otimes \omega \otimes \mu \mapsto (g \circ F) \otimes f^*\omega \otimes f^*\mu.$$

ii.) Note that with the representation $A^{p,q}(U) = A^p(U) \otimes_{\mathbb{R}} \Lambda^q \mathbb{R}^{r*}$ the pullback for an affine map F as above can be written as

$$F^*\colon A^p(U) \otimes_{\mathbb{R}} \Lambda^q \mathbb{R}^{r*} \to A^p(U') \otimes_{\mathbb{R}} \Lambda^q \mathbb{R}^{r'*} \omega \otimes_R \mu \mapsto F^*\omega \otimes_{\mathbb{R}} f^*\mu,$$

where $F^*\omega$ is the usual pullback of smooth p-forms with respect to the smooth function F. In particular we obtain the corresponding result that F^* commutes with d', d'' and d.

iii.) For $n'_1, \ldots, n'_{p+q} \in \mathbb{R}^{r'}$ and $x' \in U'$ the evaluation as in Remark 2.2 of the pullback can be written as

$$\langle F^*\alpha(x'); n'_1, \ldots, n'_{p+q}\rangle = \langle \alpha(F(x')); f(n'_1), \ldots, f(n'_{p+q})\rangle.$$

iv.) Let $F\colon U' \to U$ be a smooth map where $U \subseteq \mathbb{R}^r$ and $U' \subseteq \mathbb{R}^{r'}$ are open subsets. We can define a 'naive' pullback

$$F^*\colon A^{p,q}(U) = A^p(U) \otimes_{\mathcal{C}^\infty(U)} A^q(U) \to A^p(U') \otimes_{\mathcal{C}^\infty(U')} A^q(U') = A^{p,q}(U')$$

which is just given by the tensor product of the usual pullbacks of smooth differential p- respectively q-forms. This construction and the definition in i.) match for affine maps, however in general for smooth maps it doesn't commute with d', d'', d. Indeed, let $U = \mathbb{R}^2$ and $U' = \mathbb{R}$ and $F(x, y) = xy$ and t the coordinate in $\mathbb{R}$, then $d'F^*(d''t) = d'(yd''x + xd''y) = d'y \wedge d''x + d'x \wedge d''y \neq 0$, but $d'(d''t) = 0$ and hence $d'F^*(d''t) \neq F^*(d'(d''t))$.
The reason is that $d' = D \otimes \mathrm{id}$, but the pullback on the second factor uses the differential of F at the point $x \in \mathbb{R}^{r'}$, which might depend on x. In the affine case however, the differential has no such dependence.

3. Superforms on polyhedral complexes

3.1. Reminder of basic definitions in convex geometry.

i.) A *polyhedron* $\sigma \subseteq \mathbb{R}^r$ is the intersection of finitely many halfspaces $H_i = \{w \in \mathbb{R}^r \mid \langle u_i, w\rangle \leq c_i\}$ with $c_i \in \mathbb{R}$ and $u_i \in \mathbb{R}^{r*}$, $i \in \{1, \ldots, n\}$. A *polytope* is a bounded polyhedron.

ii.) We say that σ is an *integral* Γ*-affine polyhedron* for an additive subgroup of $\mathbb{R}$ if we may choose all $u_i \in \mathbb{Z}^{r*}$ and $c_i \in \Gamma$.

iii.) Let $J = \{j \in \{1, \ldots, n\} \mid \langle u_j, w\rangle = c_j \ \forall w \in \sigma\}$. Then $\mathbb{A}_\sigma = \{x \in \mathbb{R}^r \mid \langle u_j, x\rangle = c_j \ \forall j \in J\}$ is the smallest affine subspace of $\mathbb{R}^r$ which contains σ. Its underlying linear subspace is $\mathbb{L}_\sigma = \{x \in \mathbb{R}^r \mid \langle u_j, x\rangle = 0 \ \forall j \in J\}$. The *dimension of* σ is $\dim \sigma := \dim \mathbb{L}_\sigma$.

iv.) In particular for an integral Γ-affine polyhedron σ (recall that $\ker(A) \cap \mathbb{Z}^r$ is a lattice for a matrix A with integral coefficients) we obtain a lattice $\mathbb{Z}_\sigma := \mathbb{L}_\sigma \cap \mathbb{Z}^r$ in $\mathbb{L}_\sigma$.

v.) The *face* of a polyhedron σ is either σ itself, the empty set or an intersection of σ with the boundary of one of its defining halfspaces.

vi.) An (integral Γ-affine) *polyhedral complex* $\mathscr{C}$ in $\mathbb{R}^r$ is a finite set of (integral Γ-affine) polyhedra in $\mathbb{R}^r$ which satisfies the following conditions:
   a.) If $\sigma \in \mathscr{C}$ then all faces of σ lie in $\mathscr{C}$.
   b.) If $\sigma, \tau \in \mathscr{C}$, then $\sigma \cap \tau$ is a face of both.

vii.) The *support* $|\mathscr{C}|$ of $\mathscr{C}$ is the union of all polyhedra in $\mathscr{C}$. The polyhedral complex $\mathscr{C}$ is called *pure dimensional of dimension* n if every maximal polyhedron in $\mathscr{C}$ has dimension n. Write $\mathscr{C}_k := \{\sigma \in \mathscr{C} \mid \dim \sigma = k\}$ for $k \in \mathbb{N}$.

viii.) A polyhedral complex $\mathscr{D}$ *subdivides* the polyhedral complex $\mathscr{C}$ if they have the same support and every $\delta \in \mathscr{D}$ is contained in some $\sigma \in \mathscr{C}$. We then say $\mathscr{D}$ is a *subdivision of* $\mathscr{C}$.

3.2. **Definition.** Let $\mathscr{C}$ be a polyhedral complex in $\mathbb{R}^r$ and Ω an open subset of $|\mathscr{C}|$.

i.) A *superform* $\alpha \in A^{p,q}(\Omega)$ *of bidegree* (p,q) is given by a superform $\alpha' \in A^{p,q}(V)$ where $V \subseteq \mathbb{R}^r$ is open and $V \cap |\mathscr{C}| = \Omega$.

ii.) Two forms $\alpha' \in A^{p,q}(V)$ and $\alpha'' \in A^{p,q}(W)$ with $V \cap |\mathscr{C}| = W \cap |\mathscr{C}| = \Omega$ define the same superform in $A^{p,q}(\Omega)$ if their *restrictions to any polyhedron in* $\mathscr{C}$ *agree*. That is, for all $\sigma \in \mathscr{C}$ we have

$$\langle \alpha'(x); v_1, \dots, v_p, w_1, \dots, w_q \rangle = \langle \alpha''(x); v_1, \dots, v_p, w_1, \dots, w_q \rangle$$

for all $x \in \sigma \cap \Omega$ and $v_i, w_j \in \mathbb{L}_\sigma$.
In this case we write $\alpha'|_\sigma = \alpha''|_\sigma$. If $\alpha \in A^{p,q}(\Omega)$ is given by $\alpha' \in A^{p,q}(V)$, write $\alpha'|_\Omega = \alpha$.

3.3. **Remark.**

i.) The definition of $\wedge, d, d', d''$ on superforms on $\mathbb{R}^r$ carries over to superforms on polyhedral complexes.

ii.) Let $F\colon \mathbb{R}^{r'} \to \mathbb{R}^r$ be an affine map $F(x) = f(x) + a$ with $F(|\mathscr{C}'|) \subseteq |\mathscr{C}|$ for polyhedral complexes $\mathscr{C}' \subseteq \mathbb{R}^{r'}$ and $\mathscr{C} \subseteq \mathbb{R}^r$. Then we have $f(\mathbb{L}_{\sigma'}) \subseteq \mathbb{L}_\sigma$ for all $\sigma' \in \mathscr{C}'$ with $F(\sigma') \subseteq \sigma$ for some $\sigma \in \mathscr{C}$, after passing to some subdivision if necessary. Hence the pullback in Remark 2.4 carries over to a pullback $F^*\colon A^{p,q}(\mathscr{C}) \to A^{p,q}(\mathscr{C}')$.

4. Moment Maps and Tropical Charts

In this and the following section, K is an algebraically closed and complete field endowed with a nontrivial non-Archimedean absolute value $|\cdot|_K$ (sometimes we just write $|\cdot|$). In particular the residue field $\tilde{K}$ is also algebraically closed. Let $\nu := -\log|\cdot|$ be the associated valuation and $\Gamma := \nu(K^*) \subset \mathbb{R}$ its value group. Note that Γ is a divisible, dense subgroup of $\mathbb{R}$.

Also in the following let X always be an algebraic variety over K, i.e. an integral ($\iff$ reduced and irreducible), separated K-scheme of finite type. Furthermore note that any open subscheme of X is an algebraic variety again.

4.1. **Remark (Analytification).** We recall that the topological space of the analytification X^{an} of X is the space of all pairs $(\mathfrak{p}, p = |\cdot|_p)$, where $\mathfrak{p} \in X$ and $|\cdot|_p$ is an absolute value on the field $\kappa(\mathfrak{p}) = \mathcal{O}_{X,\mathfrak{p}}/\mathfrak{m}_{X,\mathfrak{p}}$ which induces $|\cdot|$ on K. The space X^{an} is endowed with the coarsest topology, such that the map

$$\pi = \ker : X^{\text{an}} \to X, \quad (\mathfrak{p}, p = |\cdot|_p) \mapsto \mathfrak{p}$$

is continuous and such that for each Zariski open subset U in X and each $f \in \mathcal{O}_X(U)$ the map

$$\pi^{-1}(U) \to \mathbb{R}, \quad (\mathfrak{p}, p = |\cdot|_p) \mapsto |f(p)| := |f(\mathfrak{p})|_p$$

is continuous.

Furthermore note that if X is affine, X^{an} is exactly the space of multiplicative seminorms extending $|\cdot|_K$ endowed with the usual topology.

A morphism of varieties $\varphi\colon X \to Y$ induces a morphism on the analytifications $\varphi^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$. On topological spaces, this is given locally by precomposing with $\varphi^{\#}$, i.e. if $\mathfrak{q} = \varphi(\mathfrak{p})$ for some $\mathfrak{p} \in X$, a pair $(\mathfrak{p}, |\cdot|_p)$ maps to $(\mathfrak{q}, |\cdot|_q)$, where $|\cdot|_q$ is obtained by

$$\mathcal{O}_{Y,\mathfrak{q}}/\mathfrak{m}_{Y,\mathfrak{q}} \xrightarrow{\varphi^{\#}_{\mathfrak{p}}} \mathcal{O}_{X,\mathfrak{p}}/\mathfrak{m}_{X,\mathfrak{p}} \xrightarrow{|\cdot|_p} \mathbb{R}_{\geq 0}.$$

In the affine case $\varphi\colon X = \operatorname{Spec}(B) \to Y = \operatorname{Spec}(A)$ with $\varphi = \operatorname{Spec}(f\colon A \to B)$, X^{an} (respectively Y^{an}) can be seen as the set of multiplicative seminorms on B (respectively A) extending $|\cdot|_K$ and on topological spaces we have

$$\begin{aligned} \varphi^{\mathrm{an}}\colon X^{\mathrm{an}} &\to Y^{\mathrm{an}}, \\ |\cdot|_p &\mapsto [a \mapsto |f(a)|_p]. \end{aligned}$$

4.2. **Definition.** We write $T = \mathbb{G}_m^r = \operatorname{Spec}(K[T_1^{\pm 1}, \ldots, T_r^{\pm 1}])$ for the split multiplicative torus of rank r with coordinates $T_1, \ldots, T_r$. Recall that T is an affine algebraic variety via $T \cong \operatorname{Spec}(K[T_1, \ldots, T_r, S_1, \ldots, S_r]/(T_1S_1 - 1, \ldots, T_rS_r - 1))$.

i.) We define the *tropicalization map*

$$\operatorname{trop}\colon T^{\mathrm{an}} \to \mathbb{R}^r, \quad p \mapsto (-\log|T_1(p)|, \ldots, -\log|T_r(p)|),$$

which is clearly continuous.

ii.) For a closed subvariety Y of T, we call $\operatorname{Trop}(Y) := \operatorname{trop}(Y^{\mathrm{an}})$ the *tropical variety associated with* Y.

4.3. **Remark.** For a closed subvariety Y of T of dimension n, the Bieri-Groves-Theorem says that $\operatorname{Trop}(Y)$ is a finite union of n-dimensional integral Γ-affine polyhedra in $\mathbb{R}^r$. In tropical geometry it is shown even further that $\operatorname{Trop}(Y)$ is an integral Γ-affine polyhedral complex. The structure of this complex is only determined up to subdivision, which does not matter for our constructions though.

4.4. **Definition and Remark.** Let U be an open subset of the algebraic variety X. A *moment map* is a morphism $\varphi\colon U \to T$ to some split split multiplicative torus $T = \mathbb{G}_m^r$. The *tropicalization of* φ is

$$\varphi_{\mathrm{trop}} := \operatorname{trop} \circ \varphi^{\mathrm{an}}\colon U^{\mathrm{an}} \xrightarrow{\varphi^{\mathrm{an}}} T^{\mathrm{an}} \xrightarrow{\operatorname{trop}} \mathbb{R}^r.$$

This is a continuous map with respect to the topology on U^{an}.

Let $U' \subseteq U$ be another open subset with moment map $\varphi'\colon U' \to T' = \mathbb{G}_m^{r'}$. We say that φ' *refines* φ if there exists an affine morphism of tori $\psi\colon \mathbb{G}_m^{r'} \to \mathbb{G}_m^r$, such that $\varphi = \psi \circ \varphi'$ on U'.

Here an *affine morphism of tori* stems from a group homomorphism $\mathbb{Z}^r \to \mathbb{Z}^{r'}$ composed with a (multiplicative) translation, i.e. it comes from a morphism

$$K[T_1^{\pm 1}, \ldots, T_r^{\pm 1}] \to K[T_1^{\pm 1}, \ldots, T_{r'}^{\pm 1}]$$
$$T_i \mapsto a_i T^{z_i},$$

where $a_i \in K^*$ and $z_i = (z_{i,1}, \ldots, z_{i,r'}) \in \mathbb{Z}^{r'}$ with $T^{z_i} := T_1^{z_{i,1}} \cdots T_{r'}^{z_{i,r'}}$.

Now in the situation of a refinement above let $x \in (U')^{\mathrm{an}}$ and set $c := \varphi^{\mathrm{an}}(x) \in T^{\mathrm{an}}$ respectively $c' := (\varphi')^{\mathrm{an}}(x) \in (T')^{\mathrm{an}}$. The i-th component of $\varphi_{\mathrm{trop}}(x)$ satisfies

$$\varphi_{\mathrm{trop}}(x)_i = -\log|T_i(c)| = -\log|T_i(\psi^{\mathrm{an}}(c'))| = -\log|\psi(T_i)(c')| =$$
$$= -\log|a_i T^{z_i}(c')| = -\log|a_i| + \sum_{j=1}^{r'} z_{i,j}\left(-\log|T_j(c')|\right) = -\log|a_i| + \sum_{j=1}^{r'} z_{i,j}\varphi'_{\mathrm{trop}}(x)_j.$$

Hence we see that ψ induces an integral Γ-affine map $\mathrm{Trop}(\psi)\colon \mathbb{R}^{r'} \to \mathbb{R}^r$ such that $\varphi_{\mathrm{trop}} = \mathrm{Trop}(\psi) \circ \varphi'_{\mathrm{trop}}$ on $(U')^{\mathrm{an}}$.

4.5. **Remark.** If $\varphi_i\colon U_i \to \mathbb{G}_m^{r_i}$ are finitely many moment maps of nonempty open subsets $U_1, \ldots, U_n$ of X, then $U := \bigcap_i U_i$ is an open subset of X which is nonempty (because as variety, X is irreducible). Note that the fibre product $\prod_i \mathbb{G}_m^{r_i} \cong \mathrm{Spec}(\bigotimes_i K[T_1^{\pm 1}, \ldots, T_{r_i}^{\pm 1}]) \cong \mathbb{G}_m^{\sum_i r_i}$ is a split torus as well and the universal property of the product yields a morphism

$$\varphi := \varphi_1 \times \cdots \times \varphi_n\colon U \to \mathbb{G}_m^{\sum_i r_i},$$

which refines each φ_i via the canonical projection maps. Moreover the universal property of the fibre product immediately yields that for $U' \subseteq U$ open every moment map $\varphi'\colon U' \to T'$ which refines every φ_i also refines φ.

4.6. **Lemma.** Let $\varphi\colon U \to \mathbb{G}_m^r$ be a moment map on an open subset U of X and let U' be a nonempty open subset of U. Then $\varphi_{\mathrm{trop}}((U')^{\mathrm{an}}) = \varphi_{\mathrm{trop}}(U^{\mathrm{an}})$.

Proof. [Gub16, Lemma 4.9].

4.7. **Remark.** Let $U \subseteq X$ be an open affine subset. We construct a *canonical moment map* φ_U as follows: By a generalization of Dirichlet's unit theorem the group $M_U := \mathcal{O}_X(U)^*/K^*$ is free of finite rank. Choose representatives $\varphi_1, \ldots, \varphi_r \in \mathcal{O}_X(U)^*$ of a basis, and we obtain a map

$$K[T_1^{\pm 1}, \dots, T_r^{\pm 1}] \to \mathcal{O}_X(U),$$
$$T_i \mapsto \varphi_i$$

which gives a moment map $\varphi_U \colon U \to \mathbb{G}_m^r =: T_U$. Note that this moment map is 'canonical' up to base change and multiplicative translation by elements of K^*.

4.8. **Remark.** Let $f \colon X' \to X$ be morphism of algebraic varieties over K and let $U' \subseteq X'$ and $U \subseteq X$ be open subsets with $f(U') \subseteq U$. Denote by g the composition of morphisms of rings

$$\mathcal{O}_X(U) \xrightarrow{f^{\#}(U)} \mathcal{O}_{X'}(\underbrace{f^{-1}(U)}_{\supseteq U'}) \xrightarrow{|_{U'}} \mathcal{O}_{X'}(U').$$

If $\varphi_1, \dots, \varphi_r \in \mathcal{O}_X(U)^*$ (respectively $\varphi'_1, \dots, \varphi'_{r'}$) are lifts of a basis of M_U (respectively $M_{U'}$), then $g(\varphi_i) = a_i \varphi'^{z_i}$ for $a_i \in K^*$ and $z_i \in \mathbb{Z}^{r'}$. The morphism

$$K[T_1^{\pm 1}, \dots, T_r^{\pm 1}] \to K[T_1^{\pm 1}, \dots, T_{r'}^{\pm 1}],$$
$$T_i \mapsto a_i T^{z_i}$$

gives rise to a morphism $\psi_{U,U'} \colon \mathbb{G}_m^{r'} \to \mathbb{G}_m^r$, satisfying

$$\psi_{U,U'} \circ \varphi_{U'} = \varphi_U \circ f$$

on U' (to see equality note again that $\mathbb{G}_m^r$ is affine, hence $\mathrm{Hom}_{\mathrm{Sch}}(U, \mathbb{G}_m^r) \cong \mathrm{Hom}_{\mathrm{Ring}}(K[T_1^{\pm 1}, \dots, T_r^{\pm 1}], \mathcal{O}_X(U))$).

In the case $X = X'$ and $f = \mathrm{id}$, get affine morphism of tori $\psi_{U,U'} \colon \mathbb{G}_m^{r'} \to \mathbb{G}_m^r$ such that $\psi_{U,U'} \circ \varphi_{U'} = \varphi_U$ on U'. Hence for an inclusion $U' \subseteq U$ of open subsets in X, the canonical moment map $\varphi_{U'}$ always refines φ_U in the sense of Definition 4.4.

4.9. **Definition.** An open subset U of is called *very affine* if U has a closed immersion into a split torus.

4.10. **Remark.** For an open affine subset $U \subseteq X$ the following properties are clearly equivalent:

i.) The canonical moment map φ_U is a closed embedding.
ii.) U is very affine.
iii.) $\mathcal{O}_X(U)$ is finitely generated as a K-algebra by $\mathcal{O}_X(U)^*$.

The following lemma shows that all local considerations can be done using very affine open subsets.

4.11. **Lemma.** Let X be an algebraic variety.

i.) The intersection of two very affine subsets $U \hookrightarrow \mathbb{G}_m^r, U' \hookrightarrow \mathbb{G}_m^{r'}$ of X is very affine again.

ii.) The very affine open subsets of X form a basis for the Zariski topology.

Proof. i.) As X is separated, the intersection of two affine subsets $U \cap U'$ is affine again and the canonical map $U \cap U' \to U \times U'$ is a closed immersion. The natural map $\varphi \times \varphi' \colon U \times U' \to \mathbb{G}_m^r \times \mathbb{G}_m^{r'} \cong \mathbb{G}_m^{r+r'}$ is also a closed immersion, as the corresponding map on the tensor products is surjective. Hence $U \cap U' \to \mathbb{G}_m^{r+r'}$ is a closed immersion.

ii.) Let $x \in X$ and $U \subseteq X$ be open neighborhood of x. It suffices to show that there is very affine open V around x with $V \subseteq U$. As open subschemes of varieties are varieties again, and by possibly passing to a smaller open neighborhood, we can assume that U is affine with $U = \operatorname{Spec}(A)$, where A is a K-algebra of finite type, i.e. it is of the form $A = K[T_1, \ldots, T_n]/\mathfrak{a}$ for some ideal $\mathfrak{a}$. Let $\mathfrak{p}$ denote the prime ideal of A corresponding to x and $\overline{T}_i$ the class of T_i in A. Consider the elements $f_1, \ldots, f_n \in A$ with

$$f_i = \begin{cases} \overline{T_i}, \text{ if } \overline{T_i} \neq \mathfrak{p} \\ \overline{T_i} + 1, \text{ if } \overline{T_i} \in \mathfrak{p} \end{cases} .$$

Then $V := D(f_1) \cap \cdots \cap D(f_n) = D(f_1 \cdots f_n) \subseteq U = \operatorname{Spec}(A)$ is open around x and corresponds to localization $A\left[\frac{1}{f_1 \cdots f_n}\right]$. We obtain a surjective K-algebra morphism

$$\begin{aligned} K[T_1^{\pm 1}, \ldots, T_n^{\pm 1}] &\to A\left[\frac{1}{f_1 \cdots f_n}\right], \\ T_i &\to f_i \end{aligned}$$

which gives closed immersion $V \hookrightarrow \mathbb{G}_m^n$.

□

4.12. **Remark.** On a very affine open subset, we will always use the canonical moment map $\varphi_U \colon U \to T_U := \mathbb{G}_m^r$, which is a closed immersion by Remark 4.10. We write $\operatorname{Trop}(U) := \operatorname{Trop}(\varphi_U(U)) \subseteq \mathbb{R}^r$ for the tropical variety of U in T_U. For the tropicalization map we briefly write $\operatorname{trop}_U := (\varphi_U)_{\operatorname{trop}} \colon U^{\mathrm{an}} \to \mathbb{R}^r$. Recall that φ_U is only determined up to multiplicative translation and change of basis. Hence by Definition 4.4, trop_U and $\operatorname{Trop}(U)$ are only canonical up to affine translation.

4.13. **Definition.**

i.) A *tropical chart* (V, φ_U) on X^{an} consists of an open subset V of X^{an} contained in U^{an} for a very affine open subset U of X with $V = \operatorname{trop}_U^{-1}(\Omega)$ for some open subset Ω of $\operatorname{Trop}(U)$. Note that $\operatorname{trop}_U(V) = \Omega$.

ii.) A tropical chart $(V', \varphi_{U'})$ is called a *tropical subchart* of (V, φ_U) if $V' \subseteq V$ and $U' \subseteq U$.

4.14. **Remark.** Note that the analytification of morphisms preserves immersions, i.e. if $U \subseteq U'$ as subschemes, then $U^{\text{an}} \subseteq (U')^{\text{an}}$. Hence we can talk about inclusions $(U')^{\text{an}} \subseteq U^{\text{an}} \subseteq X^{\text{an}}$ as in the definition above and about intersections as below.

4.15. **Remark.** Let $(V', \varphi_{U'})$ be a tropical subchart of (V, φ_U) with $V' = \operatorname{trop}_{U'}^{-1}(\Omega')$ respectively $V = \operatorname{trop}_U^{-1}(\Omega)$ as above. By Remark 4.8 $\varphi_{U'}$ refines φ_U and there exists affine morphism $\psi_{U,U'}$ such that $\psi_{U,U'} \circ \varphi_{U'} = \varphi_U$ on U' and hence $\operatorname{trop}_U = \operatorname{Trop}(\psi_{U,U'}) \circ \operatorname{trop}_{U'}$ on $(U')^{\text{an}}$. Obtain

$$\operatorname{Trop}(U) = \operatorname{trop}_U(U^{\text{an}}) \overset{4.6}{=} \operatorname{trop}_U((U')^{\text{an}}) =$$
$$= (\operatorname{Trop}(\psi_{U,U'}) \circ \operatorname{trop}_{U'})((U')^{\text{an}}) = \operatorname{Trop}(\psi_{U,U'})(\operatorname{Trop}(U')).$$

Hence $\operatorname{Trop}(\psi_{U,U'})$ restricts to a surjective affine map of supports of polyhedral complexes

$$\operatorname{Trop}(\psi_{U,U'})\colon \operatorname{Trop}(U') \to \operatorname{Trop}(U).$$

Furthermore this yields

$$\operatorname{Trop}(\psi_{U,U'})(\Omega') = \operatorname{Trop}(\psi_{U,U'})(\operatorname{trop}_{U'}(V')) = \operatorname{trop}_U(\underbrace{V'}_{\subseteq V}) \subseteq \Omega.$$

4.16. **Proposition.** The tropical charts on X^{an} have the following properties:

i.) For every open subset $W \subseteq X^{\text{an}}$ and every $x \in W$ there exists a tropical chart (V, φ_U) with $x \in V \subseteq W$. Furthermore, V can be chosen such that $\operatorname{trop}_U(V)$ is relatively compact in $\operatorname{Trop}(U)$.

ii.) The intersection $(V \cap V', \varphi_{U \cap U'})$ of tropical charts (V, φ_U) and $(V', \varphi_{U'})$ is a tropical subchart of both.

iii.) If (V, φ_U) is a tropical chart and if U'' is a very affine open subset of U with $V \subseteq (U'')^{\text{an}}$, then $(V, \varphi_{U''})$ is a tropical subchart of (V, φ_U).

Proof. i.) As the very affine open subsets form a basis of the Zariski topology on X and X^{an} can be obtained by glueing, we may assume that $X = \operatorname{Spec}(A)$ is a very affine scheme. A basis of X^{an} is formed by subsets of the form $V := \{x \in X^{\text{an}} \mid s_1 < |f_1(x)| < r_1, \ldots, s_k < |f_k(x)| < r_k\}$ with all $f_i \in A$ and real numbers $s_i < r_i$. We can even assume that all $s_i > 0$. Indeed, let $r > 0$. As $|K^*|$ lies dense in $\mathbb{R}_{\geq 0}$, we can find a sequence $(a_n)_{n \in \mathbb{N}}$ in K^*, such that $\lim_{n \to \infty} |a_n| = 0$ and all $|a_n| < r$, and it is easy to check using the ultrametric triangle inequality that

$$\{x \in X^{\text{an}} \mid |f(x)| \in [0, r)\} = \bigcup_{i \in \mathbb{N}} \{x \in X^{\text{an}} \mid |(f + a_i)(x)| \in (\frac{|a_i|}{2}, r)\}$$

for any $f \in A$.

Now any V of such a form lies in the analytification of the very affine open subset $U := \{x \in X \mid f_1(x) \neq 0, \ldots, f_k(x) \neq 0\}$. In order to show that (V, φ_U) is a tropical chart, it remains to show that $V = \operatorname{trop}_U^{-1}(\Omega)$ for some

open subset Ω of $\mathrm{Trop}(U)$.
For this, let $g_1, \dots, g_n \in \mathcal{O}_X(U)^* = A[\frac{1}{f_1 \cdots f_k}]^*$ be lifts of a basis of $\mathcal{O}_X(U)^*/K^*$. We can assume without loss of generality that φ_U is given by the map

$$\psi : K[T_1^{\pm 1}, \dots, T_n^{\pm 1}] \to \mathcal{O}_X(U), \quad T_i \mapsto g_i.$$

For any $j \in \{1, \dots, k\}$, there are $a_j \in K^*$ and $z_j \in \mathbb{Z}^n$, such that $f_j = a_j \cdot g^{z_j} = \psi\left(a_j T^{z_j}\right)$.

Note that $\mathrm{trop}_U(x) = (-\log(|g_1(x)|), \dots, -\log(|g_n(x)|))$ for any $x \in U^{\mathrm{an}}$ and consider for any $j \in \{1, \dots, k\}$ the continuous map

$$\begin{aligned} \alpha_j \colon \mathbb{R}^n &\to \mathbb{R}, \\ (y_1, \dots, y_n) &\mapsto |a_j| \cdot \exp\left(-\sum_{i=1}^{n} z_{j,i} y_i\right). \end{aligned}$$

See easily that

$$|f_j(x)| \in (s_j, r_j) \iff \alpha_j \circ \mathrm{trop}_U(x) \in (s_j, r_j)$$

for all $j \in \{1, \dots, k\}$ and $x \in U^{\mathrm{an}}$.

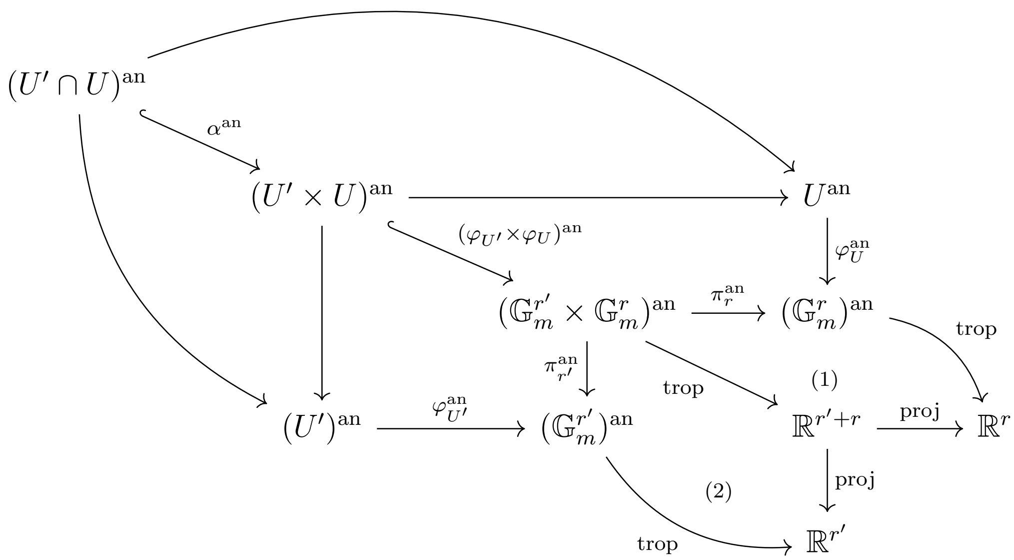
Hence $V = \mathrm{trop}_U^{-1}(\underbrace{\bigcap_{j=1}^{k} \alpha_j^{-1}(s_j, r_j)}_{=:\Omega \text{ open}})$.

Furthermore $\mathrm{trop}_U(V)$ is relatively compact, as Ω is clearly bounded, hence its closure is compact.

ii.) Let $(V, \varphi_U \colon U \to \mathbb{G}_m^r)$ respectively $(V', \varphi_{U'} \colon U' \to \mathbb{G}_m^{r'})$ be tropical charts with $\Omega = \mathrm{trop}_U(V)$ respectively $\Omega' = \mathrm{trop}_{U'}(V')$ open subsets in $\mathrm{Trop}(U)$ respectively $\mathrm{Trop}(U')$. By Lemma 4.11 the intersection $U \cap U'$ is very affine via closed embedding

$$\Phi \colon U \cap U' \xrightarrow{\alpha} U \times U' \xrightarrow{\varphi_U \times \varphi_{U'}} \mathbb{G}_m^r \times \mathbb{G}_m^{r'} \cong \mathbb{G}_m^{r+r'}.$$

Here α is the closed immersion coming from the canonical surjective map $\mathcal{O}_X(U) \otimes_K \mathcal{O}_X(U') \to \mathcal{O}_X(U \cap U')$ (X is separated, hence intersections of affines are affine). Now consider the following diagram of the underlying topological spaces of analytifications:



The commutativity of the diagram is clear except for subdiagrams (1) and (2). Show commutativity of (1) (analogously for (2)): Let $x \in (\mathbb{G}_m^{r'} \times \mathbb{G}_m^r)^{an}$, i.e. via the isomorphism as in Remark 4.5 a multiplicative seminorm in $\mathrm{Spec}(K[S_1^{\pm 1}, \ldots S_{r'}^{\pm 1}, T_1^{\pm 1}, \ldots, T_r^{\pm 1}])$. As $\pi_r^{an}(x)$ is the precomposition of x with

$$K[T_1^{\pm 1}, \ldots, T_r^{\pm 1}] \to K[S_1^{\pm 1}, \ldots S_{r'}^{\pm 1}] \otimes_K K[T_1^{\pm 1}, \ldots, T_r^{\pm 1}] \to K[S_1^{\pm 1}, \ldots S_{r'}^{\pm 1}, T_1^{\pm 1}, \ldots, T_r^{\pm 1}]$$

we see that $|T_i(\pi_r^{an}(x))| = |T_i(x)|$. Then

$$\mathrm{proj} \circ \mathrm{trop}(x) = (-\log|T_1(x)|, \ldots, -\log|T_r(x)|) = \mathrm{trop} \circ \pi_r^{an}(x).$$

Hence the whole diagram commutes. The diagram yields immediately that the set $\Omega'' := \Phi_{\mathrm{trop}}((U \cap U')^{an}) \cap (\Omega \times \Omega') \subseteq \mathbb{R}^{r+r'}$ is an open subset of $\Phi_{\mathrm{trop}}((U \cap U')^{an})$. Furthermore $\Phi_{\mathrm{trop}}^{-1}(\Omega'') = V \cap V'$. As $\varphi_{U \cap U'}$ refines Φ, we obtain affine map $\mathrm{Trop}(\psi)$ as in 4.4 such that

$$\Phi_{\mathrm{trop}} = \mathrm{Trop}(\psi) \circ \mathrm{trop}_{U \cap U'}$$

on $(U \cap U')^{an}$ which immediately yields that

$$\Omega''' := \mathrm{Trop}(\psi)^{-1}(\Omega'') \cap \mathrm{Trop}(U \cap U')$$

is an open subset of $\mathrm{Trop}(U \cap U')$ with $V \cap V' = \mathrm{trop}_{U \cap U'}^{-1}(\Omega''')$.

iii.) We need to show that $V = \mathrm{trop}_{U''}^{-1}(\Omega'')$ for some open Ω'' in $\mathrm{Trop}(U'')$. As $\varphi_{U''}$ refines φ_U, let as above $\mathrm{Trop}(\psi)$ be the affine map with $\mathrm{trop}_U = \mathrm{Trop}(\psi) \circ \mathrm{trop}_{U''}$ on $(U'')^{an}$. As (V, φ_U) is tropical chart, let $\Omega := \mathrm{trop}_U(V)$ be the corresponding open subset of $\mathrm{Trop}(U)$ with $V = \mathrm{trop}_U^{-1}(\Omega)$. From $V \subseteq (U'')^{an}$ we get as in ii.) that $V = \mathrm{trop}_{U''}^{-1}(\Omega'')$ for the open subset $\Omega'' := \mathrm{Trop}(\psi)^{-1}(\Omega) \cap \mathrm{Trop}(U'')$. □

5. Differential Forms on Algebraic Varieties

5.1. **Recollection.** A tropical chart (V, φ_U) consists of an open subset V of U^{an} for a very affine open subset U of X such that $V = \mathrm{trop}_U^{-1}(\Omega)$ for some open subset $\Omega = \mathrm{trop}_U(V)$ of $\mathrm{Trop}(U)$. Here $\varphi_U \colon U \to \mathbb{G}_m^r$ is the canonical moment map. For such a moment map we shortly write $T_U := \mathbb{G}_m^r$ and $\mathbb{R}_U := \mathbb{R}^r$ (i.e. omit the 'r').

The tropical variety $\mathrm{Trop}(U)$ is the support of a polyhedral complex in $\mathbb{R}_U$ via the tropicalization map $\mathrm{trop}_U \colon U^{\mathrm{an}} \to \mathbb{R}_U$. The canonical map φ_U is only determined up to affine morphism of tori (see Remark 4.4), hence all tropical constructions are canonical up to integral Γ-affine isomorphism.

For a tropical subchart $(V', \varphi_{U'}) \subseteq (V, \varphi_U)$ there is an affine morphism $\psi_{U,U'} \colon T_{U'} \to T_U$ with $\varphi_U = \psi_{U,U'} \circ \varphi_{U'}$ on U'.
The induced integral Γ-affine map $\mathrm{Trop}(\psi_{U,U'}) \colon \mathbb{R}_{U'} \to \mathbb{R}_U$ surjectively maps $\mathrm{Trop}(U')$ onto $\mathrm{Trop}(U)$ (see Remark 4.15) with $\mathrm{Trop}(\psi_{U,U'})(\mathrm{trop}_{U'}(V')) \subseteq \mathrm{trop}_U(V)$.

5.2. **Definition.** Consider the situation as above. We define the *restriction of a superform* $\alpha \in A^{p,q}_{\mathrm{Trop}(U)}(\Omega)$ to a superform on $\Omega' := \mathrm{trop}_{U'}(V')$ by

$$\alpha|_{V'} := \mathrm{Trop}(\psi_{U,U'})^* \alpha \in A^{p,q}_{\mathrm{Trop}(U')}(\Omega').$$

5.3. **Remark.** For tropical subcharts $(\tilde{V}, \varphi_{\tilde{U}}) \subset (V', \varphi_{U'}) \subset (V, \varphi_U)$ and $\alpha \in A^{p,q}_{\mathrm{Trop}(U)}(\Omega)$, note that

$$\mathrm{Trop}(\psi_{U,\tilde{U}}) = \mathrm{Trop}(\psi_{U,U'}) \circ \mathrm{Trop}(\psi_{U',\tilde{U}}),$$

hence

$$(\alpha|_{V'})|_{\tilde{V}} = \alpha|_{\tilde{V}}.$$

5.4. **Definition.**

i.) A *differential form* α of bidegree (p, q) on an open subset V of X^{an} is given by a family $\{(V_i, \varphi_{U_i}, \alpha_i)\}_{i \in I}$ such that
   a.) For all $i \in I$ the pair (V_i, φ_{U_i}) is a tropical chart of X^{an} and $\bigcup_{i \in I} V_i = V$.
   b.) For all $i \in I$ we have $\alpha_i \in A^{p,q}_{\mathrm{Trop}_{U_i}}(\Omega_i)$ with $\Omega_i = \mathrm{trop}_{U_i}(V_i)$.
   c.) All α_i agree on intersections, that is for all $(i, j) \in I^2$ we have

$$\alpha_i|_{V_i \cap V_j} = \alpha_j|_{V_i \cap V_j} \in A^{p,q}_{\mathrm{Trop}(U_i \cap U_j)}(\mathrm{trop}_{U_i \cap U_j}(V_i \cap V_j)).$$

ii.) If $\alpha' = \{(V_i', \varphi_{U_i'}, \alpha_i')\}_{i \in I'}$ is another differential form on V, then we consider α and α' as the same differential form if and only if

$$\alpha_i|_{V_i \cap V_j'} = \alpha_j'|_{V_i \cap V_j'}$$

for all $(i, j) \in I \times I'$.

iii.) We denote the space of (p, q)-differential forms on V by $A^{p,q}(V)$.

iv.) For $(V_i, \varphi_{U_i}, \alpha_i)$ we define the differential operator

$$d' \colon A^{p,q}(V) \to A^{p+1,q}(V)$$
$$d'\alpha := (V_i, \varphi_{U_i}, d'\alpha_i).$$

Analogously for d'', d and the wedge product $\wedge$.

5.5. **Lemma.** Let $\alpha \in A^{p,q}(V)$ be given by one canonical tropical chart (V, φ_U, α') and assume there exist tropical subcharts $\{(V_i, \varphi_{U_i})\}_{i\in I}$ of (V, φ_U) such that $\alpha'|_{V_i} = 0$ for all $i \in I$. Then already $\alpha' = 0$.

Proof. [Jel216, Lemma 3.2.12].

5.6. **Corollary.** Let $V \subseteq X^{\mathrm{an}}$ be an open subset and let $\alpha = (V_i, \varphi_{U_i}, \alpha_i)$ and $\alpha' = (V'_j, \varphi_{U'_j}, \alpha'_j)$ be two differential forms on V. Suppose there are tropical subcharts $(W_{ijl}, \varphi_{\tilde{U}_{ijl}})$ of $(V_i \cap V'_j, \varphi_{U_i \cap U'_j})$ for all i, j such that $V_i \cap V'_j = \bigcup_{ijl} W_{ijl}$ with

$$(\alpha_i|_{V_i \cap V'_j})|_{W_{ijl}} = (\alpha'_j|_{V_i \cap V'_j})|_{W_{ijl}}$$

for all l. Then $\alpha = \alpha'$.

5.7. **Remark.**

i.) Let $W \subseteq V$ be an inclusion of open subsets in X^{an} and $\alpha = \{(V_i, \varphi_{U_i}, \alpha_i)\}_{i\in I} \in A^{p,q}(V)$. By Proposition 4.16 we can choose tropical charts $\{(W_j, \varphi_{U'_j})\}_{j\in J}$ such that $\bigcup_{j\in J} W_j = W$ and for all $j \in J$ there is an $i(j) \in I$ with $W_j \subseteq V_{i(j)}$ and $U'_j \subseteq U_{i(j)}$. We then have a natural restriction map

$$\alpha|_W := (W_j, \varphi_{U'_j}, \alpha_{i(j)}|_{W_j}) \in A^{p,q}(W),$$

which is well-defined, as it is independent of the choice of tropical charts above.

Indeed, let $\{(\tilde{W}_k, \varphi_{\tilde{U}_k})\}_{k\in K}$ be another cover of W as above and let $\beta := \{(W_j, \varphi_{U'_j}, \alpha_{i(j)}|_{W_j})\}_{j\in J}$ and $\gamma := \{(\tilde{W}_k, \varphi_{\tilde{U}_k}, \alpha_{i(k)}|_{\tilde{W}_k})\}_{k\in K}$. Then for any $(j,k) \in J \times K$ have $W_j \cap \tilde{W}_k \subseteq V_{i(j)} \cap V_{i(k)}$ and hence

$$\begin{aligned} \beta_j|_{W_j\cap\tilde{W}_k} &= (\alpha_{i(j)}|_{W_j})|_{W_j\cap\tilde{W}_k} && = \alpha_{i(j)}|_{W_j\cap\tilde{W}_k} = (\alpha_{i(j)}|_{V_{i(j)}\cap V_{i(k)}})|_{W_j\cap\tilde{W}_k} \\ &= (\alpha_{i(k)}|_{V_{i(j)}\cap V_{i(k)}})|_{W_j\cap\tilde{W}_k} && = \alpha_{i(k)}|_{W_j\cap\tilde{W}_k} = (\alpha_{i(k)}|_{\tilde{W}_k})|_{W_j\cap\tilde{W}_k} \\ &&& = \gamma_k|_{W_j\cap\tilde{W}_k}. \end{aligned}$$

ii.) With the restriction map the differential forms define a presheaf $A^{p,q}(\bullet)$ on X^{an} by

$$V \mapsto A^{p,q}(V).$$

Using i.) and Corollary 5.6, we obtain that $A^{p,q}(\bullet)$ is a sheaf.

5.8. **Theorem (d''-Poincaré Lemma).** Let $V \subseteq X^{\mathrm{an}}$ be an open subset. Let $x \in V$ and $\alpha \in A^{p,q}(V)$ with $q > 0$ and $d''\alpha = 0$. Then there exists some open $W \subseteq V$ with $x \in W$ and some $\beta \in A^{p,q-1}(W)$ such that $d''\beta = \alpha|_W$.

Proof. [Jel16, Theorem 4.5].

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